`ConnectingHom`

Here we take $D = \mathbb{Z}[\sqrt{-1}]$. D is a EUCLIDean domain, which is not a field, and hence has global dimension 1. In the following example M and N will be finitely generated D-modules. For M we consider

$$M \stackrel{\varepsilon}{\to} M^{**},$$

where $M^{**} := \mathrm{Hom}_D(\mathrm{Hom}_D(M, D), D)$ and ε is the evaluation map. We also consider the short exact sequence

$$0 \to TM \stackrel{\iota}{\to} M \stackrel{\nu}{\to} M/TM \to 0,$$

where ι is the embedding and ν is the natural epimorphism. The last sequence induces via the contravariant functor $\mathrm{Hom}_D(-, N)$ the sequence

$$0 \to \mathrm{Hom}_D(M/TM, N) \xrightarrow{\mathrm{Hom}_D(\nu,N)} \mathrm{Hom}_D(M, N) \xrightarrow{\eta\ :=\ \mathrm{Hom}_D(\iota,N)} \mathrm{Hom}_D(TM, N) \to 0.$$

This sequence is again exact, since the torsion free factor M/TM over the principal ideal domain D is free and $\mathrm{Ext}^1_D(M/TM, N)$ vanishes. We start with the following diagram and only indicate the arrows we need:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & & & & & \\
 & & & & \downarrow & & & & & & \\
0 & \longleftarrow & C' & \longleftarrow & B' & \stackrel{\tau}{\longleftarrow} & A' & \longleftarrow & K' & \longleftarrow & 0 \\
 & & \downarrow \omega_1 & & \downarrow \beta_1 & & \downarrow \alpha_1 & & \downarrow \kappa_1 & & \\
0 & \longleftarrow & C & \longleftarrow & B & \stackrel{\psi}{\longleftarrow} & A & \longleftarrow & K & \longleftarrow & 0\ , \\
 & & \downarrow \omega_2 & & \downarrow \beta_2 & & \downarrow \alpha_2 & & \downarrow \kappa_2 & & \\
0 & \longleftarrow & C'' & \longleftarrow & B'' & \stackrel{\phi}{\longleftarrow} & A'' & \longleftarrow & K'' & \longleftarrow & 0 \\
 & & & & & & \downarrow & & & & \\
 & & & & & & 0 & & & &
\end{array}$$

where

$$\begin{array}{ccccccc}
B & \stackrel{\psi}{\longleftarrow} & A & & \mathrm{Hom}_D(TM, N) \oplus M^{**} & \stackrel{a\eta\oplus d\varepsilon}{\longleftarrow} & \mathrm{Hom}_D(M, N) \oplus M \\
\downarrow \beta_2 & & \downarrow \alpha_2 & = & \downarrow b\ \mathrm{Id} \oplus 0 & & \downarrow \eta \oplus \nu \\
B'' & \stackrel{\phi}{\longleftarrow} & A'' & & \mathrm{Hom}_D(TM, N) \oplus 0 & \stackrel{c\ \mathrm{Id}\oplus 0}{\longleftarrow} & \mathrm{Hom}_D(TM, N) \oplus M/TM
\end{array},$$

with $a, b, c, d \in D$ satisfying $ab = c$ for the square to be commutative. $A' \cong \mathrm{Hom}(M/TM, N) \oplus TM$ resp. B' is defined as the kernel of α_2 resp. β_2, and τ is the map induced by ψ between the kernels. The two middle columns $A' \stackrel{\alpha_1}{\to} A \stackrel{\alpha_2}{\to} A''$ and $B' \stackrel{\beta_1}{\to} B \stackrel{\beta_2}{\to} B''$ regarded as chain complexes and (τ, ψ, ϕ) as a chain map induce a kernel sequence $K' \stackrel{\kappa_1}{\to} K \stackrel{\kappa_2}{\to} K''$ and a cokernel sequence $C' \stackrel{\omega_1}{\to} C \stackrel{\omega_2}{\to} C''$. Since, as seen above, α_2 is surjective and β_1 is injective by definition, there exists a connecting homomorphism δ connecting the kernel and the cokernel sequence to a long exact sequence:

$$K' \stackrel{\kappa_1}{\to} K \stackrel{\kappa_2}{\to} K'' \stackrel{\delta}{\to} C' \stackrel{\omega_1}{\to} C \stackrel{\omega_2}{\to} C''.$$

```
> restart;
```

The package `PIR` enables one to work over several Maple-builtin principal ideal rings:

```
> with(PIR): with(homalg):
> RPP:=`PIR/homalg`;
```

$$RPP := PIR/homalg$$

Since we won't change the base ring during the computation we fix it once and for all:

```
> `homalg/default`:=RPP;
```

$$homalg/default := PIR/homalg$$

Specify $D = \mathbb{Z}[\sqrt{-1}]$, the ring of GAUSSian integers:

```
> var:=[I];
```

$$var := [I]$$

```
> Pvar(var);
```

$$[\text{“Z[I]”}]$$

Define the four variables with $c = ab$:

```
> a:=1+I; b:=5; c:=a*b; d:=2*(1+I);
```

$$a := 1 + I$$
$$b := 5$$
$$c := 5 + 5\,I$$
$$d := 2 + 2\,I$$

Define the D-module M:

```
> M:=Cokernel([[1,2,4,6],[6*(1+I)*1,6*(1+I)*3,6*(1+I)*4,6*(1+I)*5]],var
);
```

$$M := [[[1, 0, 0] = [0, 1, 0, -1], [0, 1, 0] = [0, 0, 1, 0], [0, 0, 1] = [0, 0, 0, 1]], [[6 + 6\,I, 0, 0]], \text{“Presentation”}, [6 + 6\,I, 0, 0], 2]$$

The torsion submodule TM:

```
> TM:=TorsionSubmodule(M,var);
```

$$TM := [[1 = [0, 1, 0, -1]], [6 + 6\,I], \ \text{“Presentation”}, [6 + 6\,I], 0]$$

The embedding map ι:

```
> iota:=TorsionSubmoduleEmb(M,var);
```

$$\iota := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The torsion free part $FM := M/TM$:

```
> FM:=Cokernel(iota,M,var);
```

$$FM := [[[1, 0] = [0, 0, 1, 0], [0, 1] = [0, 0, 0, 1]], [[0, 0]], \text{“Presentation”}, [0, 0], 2]$$

The natural epimorphism $M \xrightarrow{\nu} FM$:

```
> nu:=CokernelEpi(iota,M,var);
```

$$\nu := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The double dual M^{**}:

```
> HHM:=HomHom_R(M,var);
```

$$HHM := [[[1, 0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [0, 1] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}], [[0, 0]], \text{“Presentation”}, [0, 0], 2]$$

The evaluation map $M \xrightarrow{\varepsilon} M^{**}$:

```
> epsilon:=NatTrIdToHomHom_R(M,var);
```

$$\varepsilon := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The D-module N:

```
> N:=Cokernel([[1,2,4,0],[2*(1-I)*1,2*(1-I)*3,2*(1-I)*4,0],[0,0,0,2]],v
ar);
```

$$N := [[[1, 0, 0] = [0, 0, 0, 1], [0, 1, 0] = [0, -1, 0, 1], [0, 0, 1] = [0, 0, 1, 0]], [[2, 0, 0], [0, 2 + 2\,I, 0]], \text{“Presentation”}, [2, 2 + 2\,I, 0], 1]$$

The module of homomorphisms $\mathrm{Hom}_D(M,N)$:

```
> HMN:=Hom(M,N,var);
```

$$HMN := \left[\left[[1,0,0,0,0,0,0,0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, [0,1,0,0,0,0,0,0] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \right.\right.$$
$$[0,0,1,0,0,0,0,0] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, [0,0,0,1,0,0,0,0] = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 0 & 0 \\ -3 & 1 & 0 \end{bmatrix},$$
$$[0,0,0,0,1,0,0,0] = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 1 & 0 \end{bmatrix}, [0,0,0,0,0,1,0,0] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ -5 & 1 & 0 \end{bmatrix},$$
$$\left.[0,0,0,0,0,0,1,0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, [0,0,0,0,0,0,0,1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right], [$$
$$[2,0,0,0,0,0,0,0], [0,2,0,0,0,0,0,0], [0,0,2,0,0,0,0,0],$$
$$[0,0,0,2+2I,0,0,0,0], [0,0,0,0,2+2I,0,0,0], [0,0,0,0,0,2+2I,0,0]],$$
$$\left.\text{“Presentation”}, [2,2,2,2+2I,2+2I,2+2I,0,0], 2\right]$$

The module of homomorphisms $\mathrm{Hom}_D(TM,N)$:

```
> HTMN:=Hom(TM,N,var);
```

$$HTMN := [[[1,0] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, [0,1] = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}], [[2,0],[0,2+2I]], \text{“Presentation”}, [2,2+2I], 0]$$

The identity $\mathrm{Id}_{\mathrm{Hom}_D(TM,N)}$ map of $\mathrm{Hom}_D(TM,N)$:

```
> Id:=IdentityMap(HTMN,var);
```

$$Id := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The induced map $\eta := \mathrm{Hom}_D(\iota, N)$:

```
> eta:=HomMap(TM,iota,M,N,var);
```

$$\eta := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The zero module:

```
> Z:=ZeroModule(var);
```

$$Z := [[1=0],[1], \text{“Presentation”}, [1], 0]$$

The zero map from M/TM to the zero module:

```
> zeta:=ZeroMap(FM,Z,var);
```

$$\zeta := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The zero map from M^{**} to the zero module:

```
> chi:=ZeroMap(HHM,Z,var);
```

$$\chi := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A
is the direct sum of $\mathrm{Hom}_D(M, N)$ and M:

```
> A:=DirectSum(HMN,M,var);
```

$$A := [[[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] = [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] = [0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] = [1, -2, 1, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] = [-1, 3, -3, 0, 0, 1, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] = [0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] = [0, 0, 0, 1, -2, 1, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] = [0, 0, 0, -1, 3, -3, 0, 0, 1, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] = [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]], [$$
$$[2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 2 + 2\,I, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 2 + 2\,I, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 2 + 2\,I, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 6 + 6\,I, 0, 0, 0, 0]], \text{“Presentation”},$$
$$[2, 2, 2, 2 + 2\,I, 2 + 2\,I, 2 + 2\,I, 6 + 6\,I, 0, 0, 0, 0], 4]$$

A'' is the direct sum of $\mathrm{Hom}_D(TM, N)$ and M:

```
> _A:=DirectSum(HTMN,FM,var);
```

$$\_A := [[[1, 0, 0, 0] = [1, 0, 0, 0], [0, 1, 0, 0] = [-1, 1, 0, 0], [0, 0, 1, 0] = [0, 0, 1, 0],$$
$$[0, 0, 0, 1] = [0, 0, 0, 1]], [[2, 0, 0, 0], [0, 2 + 2\,I, 0, 0]], \text{“Presentation”},$$
$$[2, 2 + 2\,I, 0, 0], 2]$$

α_2 is the direct sum of the maps η and ν:

```
> alpha2:=DirectSumMap(HMN,M,eta,nu,HTMN,FM,var);
```

$$\alpha 2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A' is the kernel of α_2:

```
> A_:=Kernel(A,alpha2,_A,var);
```

$$A\_ := [[[1,0,0,0,0,0,0] = [-1,2,-2-2I,2+2I,-4-4I,2+2I,0,0,0,0,0], [0,1,0,0,0,0,0] = [-3,5,-2-2I,2+2I,-4-4I,2+2I,0,0,0,0,0], [0,0,1,0,0,0,0] = [5,-10,8+4I,-9-4I,21+8I,-16-4I,0,0,3,0,0], [0,0,0,1,0,0,0] = [1,-3,4,-4,12,-12,0,0,3,0,0], [0,0,0,0,1,0,0] = [-3,9,-11,12,-33,32,0,0,-8,0,0], [0,0,0,0,0,1,0] = [0,0,0,0,0,0,1,0,0,0], [0,0,0,0,0,0,1] = [0,0,0,0,0,0,1,0,0,0,0]], [[2,0,0,0,0,0,0], [0,2,0,0,0,0,0], [0,0,2+2I,0,0,0,0], [0,0,0,2+2I,0,0,0], [0,0,0,0,6+6I,0,0]], \text{“Presentation”}, [2,2,2+2I,2+2I,6+6I,0,0], 2]$$

α_1 is the embedding map:

```
> alpha1:=KernelEmb(A,alpha2,_A,var);
```

$$\alpha 1 := \begin{bmatrix} -1-2I & 0 & -1 & 0 & 0 & 2+2I & 0 & 0 & 0 & 0 & 0 \\ -2I & 1 & -3 & 0 & 0 & 2+2I & 0 & 0 & 0 & 0 & 0 \\ 2+4I & -1 & 4 & -1 & 0 & -6-4I & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 3 & 1 & 4 & -8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The A-sequence is exact:

```
> IsShortExactSeq(A_,alpha1,A,alpha2,_A,var,"VERBOSE");
```

$$true$$

B is the direct sum of $\mathrm{Hom}_D(TM, N)$ and M^{**}:

```
> B:=DirectSum(HTMN,HHM,var);
```

$$B := [[[1,0,0,0] = [1,0,0,0], [0,1,0,0] = [-1,1,0,0], [0,0,1,0] = [0,0,1,0], [0,0,0,1] = [0,0,0,1]], [[2,0,0,0], [0,2+2I,0,0]], \text{“Presentation”}, [2,2+2I,0,0], 2]$$

B'' is the direct sum of $\mathrm{Hom}_D(TM, N)$ and the zero module:

```
> _B:=DirectSum(HTMN,Z,var);
```

$$\_B := [[[1,0] = [-1,0,1], [0,1] = [0,-1,1]], [[2,0], [0,2+2I]], \text{“Presentation”}, [2,2+2I], 0]$$

β_2 is the direct sum of the map $b\mathrm{Id}$ and the zero map χ:

```
> beta2:=DirectSumMap(HTMN,HHM,MulMat(b,Id,var),chi,HTMN,Z,var);
```

$$\beta 2 := \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

B' is the kernel of β_2:

```
> B_:=Kernel(B,beta2,_B,var);
```

$$B\_ := [[[1,0] = [0,0,1,0], [0,1] = [0,0,0,1]], [[0,0]], \text{“Presentation”}, [0,0], 2]$$

β_1 is the embedding map:

```
> beta1:=KernelEmb(B,beta2,_B,var);
```

$$\beta 1 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The B-sequence is in this example (depending on the choice of the number b) exact:

```
> IsShortExactSeq(B_,beta1,B,beta2,_B,var,"VERBOSE");
```

$$true$$

ψ is the direct sum of the maps $a\eta$ and $d\epsilon$:

```
>  psi:=DirectSumMap(HMN,M,MulMat(a,eta,var),MulMat(d,epsilon,var),HTMN,
HHM,var);
```

$$\psi := \begin{bmatrix} 1+I & 0 & 0 & 0 \\ 1+I & 0 & 0 & 0 \\ 1+I & 0 & 0 & 0 \\ -1-I & 1+I & 0 & 0 \\ 1+I & 1+I & 0 & 0 \\ 1+I & 1+I & 0 & 0 \\ 0 & -1-I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2+2\,I & 0 \\ 0 & 0 & 0 & 2+2\,I \end{bmatrix}$$

Some infos about ψ:

```
>  IsHom(A,psi,B,var);
```

true

```
>  IsSurjective(psi,B,var);
```

false

```
>  IsInjective(A,psi,B,var);
```

false

A necessary condition to be able to complete the square:

```
>  CheckKerSq(A,alpha2,_A,psi,B,beta2,_B,var);
```

$$[\%1,\ \%1,\ \%1,\ \%1,\ \%1,\ \%1,\ \%1]$$
$$\%1 := [0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]$$

Completing the square by ϕ, which is the direct sum of the map $c\mathrm{Id}$ and the zero map ζ:

```
>  phi:=DirectSumMap(HTMN,FM,MulMat(c,Id,var),zeta,HTMN,Z,var);
```

$$\phi := \begin{bmatrix} -1-I & 0 \\ 1+I & -1-I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Some infos about ϕ:

```
>  IsHom(_A,phi,_B,var);
```

true

```
>  IsSurjective(phi,_B,var);
```

false

```
>  IsInjective(_A,phi,_B,var);
```

false

Check the commutativity of the square:

```
>  IsCommutativeSq(alpha2,phi,psi,beta2,_B,var);
```

true

The induced kernel map τ:

```
>  tau:=KernelMap(A,alpha2,_A,psi,B,beta2,_B,var);
```

$$\tau := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Some infos about τ:

```
> IsHom(A_,tau,B_,var);
```

$$\mathit{true}$$

```
> IsSurjective(tau,B_,var);
```

$$\mathit{false}$$

```
> IsInjective(A_,tau,B_,var);
```

$$\mathit{false}$$

Check the commutativity of the square:

```
> IsCommutativeSq(alpha1,psi,tau,beta1,B,var);
```

$$\mathit{true}$$

Compute the kernel sequence:

```
> K:=Kernel(A,psi,B,var);
```

$$K := [[[1, 0, 0, 0, 0, 0, 0, 0, 0] = [0, 0, 2, -2, 4, -2, 0, 0, 0, 0, 0],$$
$$[0, 1, 0, 0, 0, 0, 0, 0, 0] = [-1 - I, 2 + 2\,I, 1 - I, -2, 4, -2, 0, 0, 0, 0, 0],$$
$$[0, 0, 1, 0, 0, 0, 0, 0, 0] = [-1, 2, 2, -2, 4, -2, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 1, 0, 0, 0, 0, 0] = [-3, 5, 2, -2, 4, -2, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 1, 0, 0, 0, 0] = [0, 0, 2, -1, 3, -2, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 1, 0, 0, 0] = [-1, 3, -1, 1, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 1, 0, 0] = [6 + I, -12 - 2\,I, -7 + I, 8, -20, 12, 0, 0, -1, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 1, 0] = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 0, 0, 1] = [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0]], [$$
$$[1 + I, 0, 0, 0, 0, 0, 0, 0, 0], [0, 1 + I, 0, 0, 0, 0, 0, 0, 0], [0, 0, 2, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 2, 0, 0, 0, 0, 0], [0, 0, 0, 0, 2 + 2\,I, 0, 0, 0, 0], [0, 0, 0, 0, 0, 2 + 2\,I, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 6 + 6\,I, 0, 0]], \text{ “Presentation”},$$
$$[1 + I, 1 + I, 2, 2, 2 + 2\,I, 2 + 2\,I, 6 + 6\,I, 0, 0], 2]$$

```
> K_:=Kernel(A_,tau,B_,var);
```

$$K\_ := [[[1, 0, 0, 0, 0, 0, 0] = [-3, 5, -2 - 2\,I, 2 + 2\,I, -4 - 4\,I, 2 + 2\,I, 0, 0, 0, 0, 0],$$
$$[0, 1, 0, 0, 0, 0, 0] = [-2, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0],$$
$$[0, 0, 1, 0, 0, 0, 0] = [6, -11, 6 + 2\,I, -6 - 2\,I, 16 + 4\,I, -14 - 2\,I, 0, 0, 3, 0, 0],$$
$$[0, 0, 0, 1, 0, 0, 0] = [-4, 7, -4 - 4\,I, 5 + 4\,I, -9 - 8\,I, 4 + 4\,I, 0, 0, 0, 0, 0],$$
$$[0, 0, 0, 0, 1, 0, 0] = [0, 5, -11 + 4\,I, 11 - 4\,I, -36 + 8\,I, 40 - 4\,I, 0, 0, -11, 0, 0],$$
$$[0, 0, 0, 0, 0, 1, 0] = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0],$$
$$[0, 0, 0, 0, 0, 0, 1] = [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0]], [[2, 0, 0, 0, 0, 0, 0],$$
$$[0, 2, 0, 0, 0, 0, 0], [0, 0, 2 + 2\,I, 0, 0, 0, 0], [0, 0, 0, 2 + 2\,I, 0, 0, 0],$$
$$[0, 0, 0, 0, 6 + 6\,I, 0, 0]], \text{ “Presentation”}, [2, 2, 2 + 2\,I, 2 + 2\,I, 6 + 6\,I, 0, 0], 2]$$

```
> kappa1:=KernelMap(A_,tau,B_,alpha1,A,psi,B,var);
```

$$\kappa 1 := \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 1 & 0 & 0 & -3 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> _K:=Kernel(_A,phi,_B,var,"var_to_assign_embedding_info"='_KK');
```

$\_K := [[[1, 0, 0, 0] = [-2, 2, 0, 0], [0, 1, 0, 0] = [-3 - I, 2, 0, 0], [0, 0, 1, 0] = [0, 0, 1, 0], [0, 0, 0, 1] = [0, 0, 0, 1]], [[1 + I, 0, 0, 0], [0, 1 + I, 0, 0]], \text{“Presentation”}, [1 + I, 1 + I, 0, 0], 2]$

```
> copy(_KK);
```

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 - I & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> kappa2:=KernelMap(A,psi,B,alpha2,_A,phi,_B,var);
```

$$\kappa 2 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The kernel sequence has a non-zero cokernel at K'':

```
> IsShortExactSeq(K_,kappa1,K,kappa2,_K,var,"VERBOSE");
```

$\text{“homs”} = \mathit{true}, \text{“cmps”} = \mathit{true}, \text{“defs”} = [\mathit{true}, \mathit{true}, [[[1, 0] = [0, 0, 1, 0], [0, 1] = [0, 0, 0, 1]], [[0, 0]], \text{“Presentation”}, [0, 0], 2]]$

Define the cokernel sequence:

```
> C:=Cokernel(psi,B,var);
```

$C := [[[1, 0, 0, 0] = [-1, 1, 0, 0], [0, 1, 0, 0] = [-2, 1, 0, 0], [0, 0, 1, 0] = [3, -2, 0, 1], [0, 0, 0, 1] = [0, 0, -1, 1]], [[1 + I, 0, 0, 0], [0, 1 + I, 0, 0], [0, 0, 2 + 2\,I, 0], [0, 0, 0, 2 + 2\,I]], \text{“Presentation”}, [1 + I, 1 + I, 2 + 2\,I, 2 + 2\,I], 0]$

```
> C_:=Cokernel(tau,B_,var,"var_to_assign_embedding_info"='CC_');
```

$C\_ := [[[1, 0] = [0, 0, 1, 0], [0, 1] = [0, 0, 0, 1]], [[0, 0]], \text{“Presentation”}, [0, 0], 2]$

```
> copy(CC_);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> omega1:=CokernelMap(tau,B_,beta1,psi,B,var);
```

$$\omega 1 := \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

```
> _C:=Cokernel(phi,_B,var);
```

$\_C := [[[1, 0] = [0, -1, 1], [0, 1] = [1, -1, 0]], [[1 + I, 0], [0, 1 + I]],$ "Presentation" ,
$[1 + I, 1 + I], 0]$

```
> omega2:=CokernelMap(psi,B,beta2,phi,_B,var);
```

$$\omega 2 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$$

The cokernel sequence has a non-zero kernel at C':

```
> IsShortExactSeq(C_,omega1,C,omega2,_C,var,"VERBOSE");
```

"homs" = *true*, "cmps" = *true*, "defs" = [[
$[[1, 0] = [0, 0, 2 + 2I, 0], [0, 1] = [0, 0, 0, 2 + 2I]], [[0, 0]],$ "Presentation", $[0, 0],$
$2]$, *true*, *true*]

Compute the connecting homomorphism between the kernel and the cokernel sequence:

```
> delta:=ConnectingHom(_K,alpha2,psi,tau,beta1,C_,var,"Hqn_embedding_in
fo"=_KK,"Hsn_1_embedding_info"=CC_,"Cqn_Bqn"=_A,"Zn_1"=B,"Zsn_1"=B_);
```

$$\delta := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 + 2I & 0 \\ 0 & 2 + 2I \end{bmatrix}$$

The resulting sequence is a long exact sequence:

```
> IsExactCoseq([K_,kappa1,K,kappa2,_K,delta,C_,omega1,C,omega2,_C],var,
"VERBOSE");
```

true

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References

[Bar07] Mohamed Barakat, PIR: *A tiny* homalg *ring package for* Maple*-built-in principal ideal rings*, 2004-2007, (http://wwwb.math.rwth-aachen.de:8040).

[BR] Mohamed Barakat and Daniel Robertz, homalg – *A meta-package for homological algebra*, submitted. arXiv:math.AC/0701146 and (http://wwwb.math.rwth-aachen.de/homalg).

[BR07] ———, homalg *project*, 2004-2007, (http://wwwb.math.rwth-aachen.de/homalg).

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