BC_2

```
> restart:
> with(Involutive): with(homalg):
```

Tell `Involutive` to compute over the integers:

```
> InvolutiveOptions("rational",false);
```

$$true$$

```
> RPI:=`Involutive/homalg`;
```

$$RPI := Involutive/homalg$$

```
> `homalg/default`:=RPI;
```

$$homalg/default := Involutive/homalg$$

The cyclic group $C_2 = \mathrm{O}(1)$:
The group ring $\mathbb{Z}C_2 = \mathbb{Z}[x]/\langle x^2 - 1\rangle$:

```
> var:=[[x],[x^2-1]];
```

$$var := [[x], [x^2 - 1]]$$

The trivial $\mathbb{Z}C_2$-module $\mathbb{Z}$:

```
> Z:=[x-1];
```

$$Z := [x - 1]$$

THEOREM: *For a discrete group G there are natural isomorphisms*

$$\mathrm{Ext}^{\bullet}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) = \mathrm{H}^{\bullet}(G,\mathbb{Z}) = \mathrm{H}^{\bullet}(\mathrm{B}G;\mathbb{Z}).$$

The integral cohomology of the group C_2 = integral cohomology of the classifing space

$$\mathrm{B}C_2 = S^{\infty}/C_2 = \mathbb{R}P^{\infty} = \mathrm{Gr}_1^{\mathbb{R}}(\infty) = \mathrm{BGL}(1,\mathbb{R}) = \mathrm{BO}(1).$$

(Cf. [LM89, Proposition B.5, p. 378 and p. 379])
The zeroth integral cohomology $\mathrm{H}^0(\mathbb{R}P^{\infty};\mathbb{Z}) = \mathbb{Z}$:

```
> Ext(0,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]], [x - 1], \text{“Presentation”}]$$

The first integral cohomology $\mathrm{H}^1(\mathbb{R}P^{\infty};\mathbb{Z}) = 0$:

```
> Ext(1,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 0 \end{array}\right]], [1], \text{“Presentation”}]$$

The second integral cohomology $\mathrm{H}^2(\mathbb{R}P^{\infty};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$:

```
> Ext(2,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]], [2, x - 1], \text{“Presentation”}]$$

The third integral cohomology $\mathrm{H}^3(\mathbb{R}P^{\infty};\mathbb{Z}) = 0$:

```
> Ext(3,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 0 \end{array}\right]], [1], \text{“Presentation”}]$$

The fourth integral cohomology $\mathrm{H}^2(\mathbb{R}P^{\infty};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$:

```
> Ext(4,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]], [2, x - 1], \text{“Presentation”}]$$

The fifth integral cohomology $\mathrm{H}^5(\mathbb{R}P^{\infty};\mathbb{Z}) = 0$:

```
> Ext(5,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 0 \end{array}\right]], [1], \text{“Presentation”}]$$

etc...

```
> Ext(6,Z,Z,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]], [2, x - 1], \text{“Presentation”}]$$

The trivial $\mathbb{Z}C_2$-module $\mathbb{Z}/2\mathbb{Z}$:

```
> Z2:=[2,x-1];
```

$$Z2 := [2,\ x-1]$$

THEOREM [LM89, Theorem B.7, p. 380]: *The cohomology ring of C_2 is a $\mathbb{Z}/2\mathbb{Z}$-polynomial ring*

$$\mathrm{Ext}^{\bullet}_{\mathbb{Z}C_2}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^{\bullet}(\mathrm{B}C_2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1],$$

where w_1 is the first STIEFEL-WHITNEY *class.*

```
> Ext(0,Z,Z2,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]],\ [2,\ x-1],\ \text{“Presentation”}]$$

```
> Ext(1,Z,Z2,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]],\ [2,\ x-1],\ \text{“Presentation”}]$$

```
> Ext(2,Z,Z2,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]],\ [2,\ x-1],\ \text{“Presentation”}]$$

```
> Ext(3,Z,Z2,var);
```

$$[[1 = \left[\begin{array}{c} 1 \end{array}\right]],\ [2,\ x-1],\ \text{“Presentation”}]$$

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