# The Homomorphism Theorem and Effective Computations

Habilitationsschrift vorgelegt von

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# Preface

I would like to thank all people who were part of my private and academic life in the past years.

There are no words I can use to express my gratitude to my teacher Prof. Wilhelm Plesken: I will simply say thank you for all what you taught me and all the wonderful mathematical and private discussions we had.

Also my thanks to those who are always supporting me secretly, especially Prof. Vladimir Gerdt, Prof. Sebastian Walcher, and Dr. Alban Quadrat. There are probably more secret supporters out there I didn't discover yet. Also to them I would like to say thank you.

I will never forget the wonderful years with my colleagues and friends Thomas Bächler, Barbara Bremer, Anna Fabianska, Sabrina Grande, Gehrt Hartjen, Dagmar van Heiss, Sebastian Jambor, Jan Jongen, Markus Kirschmer, Wolfgang Krass, Markus Lange-Hegermann, Arne Lorenz, Daniel Robertz, Tilman Schulz, and my special friend Simon Görtzen at the Lehrstuhl B für Mathematik.

My deep love to my parents and sister for all what they gave me back then and now.

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This work is dedicated to my family, Miriam, Josef, Jonas, and my wife Irene. Thank you for your unbounded support, unparalleled patience, and infinite love.

# Introduction

This thesis contains two parts, which have two things in common. The first of them is the desire to push abstract theory to the point where things become concrete; so concrete that a computer[1] can understand them. Computers are so stupid that they cannot make sense of the widely used statement "details are left to the reader". The second thing the two parts have in common is the extensive use they make of a computational beast called spectral sequences. I hope that after the lecture of this thesis the reader will be convinced that spectral sequences are nothing but the homomorphism theorem, only doing its best to look scary. In my attempt to tame the beast a software project called `homalg` was born [**ht09**]. It took eleven mathematicians from Aachen, myself included, over a year to develop this project. One of the several cores of this project is a package, also called `homalg` [**Bar09**], which, as of writing these lines, contains more than 40000 lines of GAP code excluding documentation. It was built following GROTHENDIECK's rising sea philosophy until spectral sequences (of bicomplexes) got quietly flooded.

The current state of the documentation of the `homalg` package can be found after page 83. It is still work in progress.

In the first part of the thesis (Chapter 1 + Appendix) spectral sequences are used as a computational tool. In the second part they are used in a theoretical way exactly like the homomorphism theorem, namely to transfer questions about one structure to question about another where they suddenly become a lot simpler to answer.

[1]Equipped with GAP ;)

# Contents

# Spectral Filtrations via Generalized Morphisms

## 1. Introduction

The motivation behind this work was the need for algorithms to explicitly construct several natural filtrations of modules. It is already known that all these filtrations can be described in a unified way using spectral sequences of filtered complexes, which in turn suggests a unified algorithm to construct all of them. Describing this algorithm is the main objective of the present paper.

Since VERDIER it became more and more apparent that one should be studying complexes of modules rather than single modules. A single module is then represented by one of its resolutions, all quasi-isomorphic to each other. The idea is now very simple:

> If there is no direct way to construct a certain natural filtration on a module $M$, it might be simpler to explicitly realize $M$ as one of the (co)homologies $H_n(C)$ of some complex $C$ with some easy constructible (natural) filtration, such that the filtration induced on $H_n(C)$ (by the one on $C$) maps by the explicit isomorphism $H_n(C) \cong M$ onto the looked-for filtration on $M$.

In this work it will be shown how to compute the induced filtration on $H_n(C)$ using spectral sequences of filtered complexes, enriched with some extra data. This provides a unified approach for constructing numerous important filtrations of modules and sheaves of modules (cf. [**Wei94**, Chap. 5] and [**Rot79**, Chap. 11]). Since we are interested in effective computations we restrict ourself for simplicity to *finite type* complexes carrying *finite* filtrations.

When talking about $D$-modules the ring $D$ is assumed associative with one.

**Definition 1.1** (Filtered module)**.** Let $M$ be a $D$-module.

(a) A chain of submodules $(F_pM)_{p\in\mathbb{Z}}$ of the module $M$ is called an **ascending filtration** if $F_{p-1}M \leq F_pM$. The $p$-th **graded part** is the subfactor module defined by $\operatorname{gr}_p M := F_pM/F_{p-1}M$.

(d) A chain of submodules $(F^pM)_{p\in\mathbb{Z}}$ of the module $M$ is called a **descending filtration** if $F^pM \geq F^{p+1}M$. The $p$-th **graded part** is the subfactor module defined by $\operatorname{gr}^p M := F^pM/F^{p+1}M$.

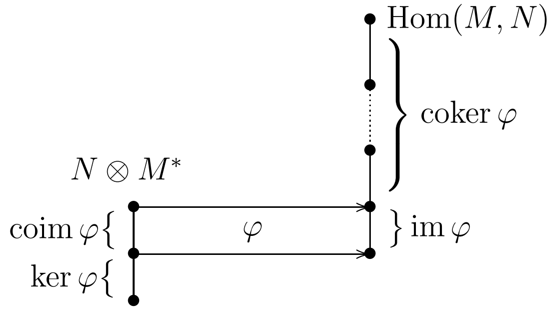
All filtrations of modules will be assumed **exhaustive** (i.e. $\bigcup_p F_pM = M$), **Hausdorff** (i.e. $\bigcap_p F_pM = 0$), and will have **finite length** $m$ (i.e. the difference between the highest and the lowest stable index is at most $m$). Such filtrations are called $m$-step filtrations.

We start with two examples that will be pursued in Section 9:

**(d)** Let $M$ and $N$ be right $D$-modules and $M^* := \mathrm{Hom}_D(M, D)$ the dual (left) $D$-module of $M$. The map

$$\varphi : \begin{cases} N \otimes_D M^* & \to & \mathrm{Hom}_D(M, N) \\ n \otimes \alpha & \mapsto & (m \mapsto n\alpha(m)) \end{cases}$$

is in general neither injective nor surjective. In fact, $\mathrm{im}\,\varphi$ is the last (graded) part of a **d**escending filtration of $\mathrm{Hom}(M, N)$.



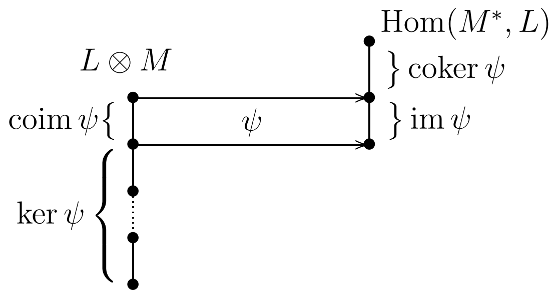

**(a)** Dually, let $M$ be a left module, $L$ a right module, and

$$\varepsilon : M \to M^{**} := \mathrm{Hom}(\mathrm{Hom}(M, D), D)$$

the **evaluation map**. The composition $\psi$

$$\begin{tikzcd} L \otimes_D M \arrow[r, "\mathrm{id}\otimes\varepsilon"] \arrow[rr, bend right, "\psi"'] & L \otimes M^{**} \arrow[r, "\varphi"] & \mathrm{Hom}_D(M^*, L) \end{tikzcd}$$

is in general neither injective nor surjective. It will turn out that its coimage $\mathrm{coim}\,\psi$ is the last graded part of an **a**scending filtration of $L \otimes M$.




Example **(a)** has a geometric interpretation.

**(a')** Let $D$ be a commutative NOETHERian ring with 1. Recall that the KRULL dimension $\dim D$ is defined to be the length $d$ of a maximal chain of prime ideals $D > \mathfrak{p}_0 > \cdots > \mathfrak{p}_d$. For example, the KRULL **dimension** of a field $k$ is zero, $\dim \mathbb{Z} = 1$, and $\dim D[x_1, \ldots, x_n] = \dim D + n$.

The definition of the KRULL dimension is then extended to nontrivial $D$-modules using

$$\dim M := \dim \frac{D}{\mathrm{Ann}_D(M)}.$$

Define the **codimension** of a nontrivial module $M$ as

$$\operatorname{codim} M := \dim D - \dim M$$

and set the codimension of the zero module to be $\infty$. If for example $D$ is a (commutative) principal ideal domain which is not a field, then the finitely generated $D$-modules of codimension 1 are precisely the finitely generated torsion modules.

**Definition 1.2** (Purity filtration)**.** Let $D$ be a commutative NOETHERian ring with 1 and $M$ a $D$-module. Define the submodule $\mathrm{t}_{-c} M$ as the biggest submodule of $M$ of codimension $\geq c$. The *ascending* filtration

$$\cdots \leq \mathrm{t}_{-(c+1)} M \leq \mathrm{t}_{-c} M \leq \cdots \leq \mathrm{t}_{-1} M \leq \mathrm{t}_0 M := M$$

is called the **purity filtration** of $M$ [**HL97**, Def. 1.1.4]. The graded part $M_c := \mathrm{t}_{-c} / \mathrm{t}_{-(c+1)}$ is **pure** of codimension $c$, i.e. any nontrivial submodule of $M_c$ has codimension $c$. $\mathrm{t}_{-1} M$ is nothing but the torsion submodule $\mathrm{t}(M)$. This suggests calling $\mathrm{t}_{-c} M$ the **$c$-th (higher) torsion submodule** of $M$.

Early references to the purity filtration are J.-E. ROOS's pioneering paper [**Roo62**] where he introduced the **bidualizing complex**, M. KASHIWARA's master thesis (December 1970) [**Kas95**, Theorem 3.2.5] on algebraic $D$-modules, and J.-E. Björk's standard reference [**Bjö79**, Chap. 2, Thm. 4.15]. All these references address the construction of this filtration from a homological[2] point of view, where the assumption of commutativity of the ring $D$ can be dropped.

Under some mild conditions on the *not* necessarily commutative ring $D$ one can characterize the purity filtration in the following way: There exist so-called **higher evaluation maps** $\varepsilon_c$, generalizing the standard evaluation map, such that the sequence

$$0 \to \mathrm{t}_{-(c+1)} M \to \mathrm{t}_{-c} M \xrightarrow{\varepsilon_c} \operatorname{Ext}^c_D(\operatorname{Ext}^c_D(M, D), D)$$

is exact (cf. [**AB69, Qua01**]). $\varepsilon_c$ can thus be viewed as a **natural transformation** between the **$c$-th torsion functor** $\mathrm{t}_{-c}$ and the **$c$-th bidualizing functor** $\operatorname{Ext}^c(\operatorname{Ext}^c(-, D), D)$. In Subsection 9.1.3 it will be shown how to use spectral sequences of filtered complexes to construct all the higher evaluation maps $\varepsilon_c$. More generally it is evident that spectral sequences are natural birthplaces for many natural transformations.

Now to see the connection to the previous example (a) set $L = D$ as a right $D$-module. $\psi$ then becomes the evaluation map $\varepsilon$.

There still exists a misunderstanding concerning spectral sequences of filtered complexes and it might be appropriate to address it here. Let $C$ be a filtered complex (cf. Def. 3.1 and Remark 4.6). (*) We even assume $C$ of *finite type* and the filtration *finite*. The filtration on $C$ *induces* a filtration on its (co)homologies $H_n(C)$. It is sometimes believed that the spectral sequence $E^r_{pq}$ associated to the filtered complex $C$ cannot be used to determine the

[2] KASHIWARA did not use spectral sequences: "Instead of using spectral sequences, Sato devised [...] a method using associated cohomology", [**Kas95**, Section 3.2].

induced filtration on $H_n(C)$, but can only be used to determine its graded parts $\operatorname{gr}_p H_n(C)$. One might be easily led to this conclusion since the last page of the spectral sequence consists of precisely these graded parts $E^\infty_{pq} = \operatorname{gr}_p H_{p+q}(C)$, and computing the last page is traditionally regarded as the last step in determining the spectral sequence. It is clear that even the knowledge of the total (co)homology $H_n(C)$ as a whole (along with the knowledge of the graded parts $\operatorname{gr}_p H_n(C)$) is in general *not* enough to determine the filtration. Another reason might be the use of the phrase "computing a spectral sequence". Very often this means a successful attempt to figure out the morphisms on some of the pages of the spectral sequence, or even better, working skillfully around determining most or even all of these morphisms and nevertheless deducing enough or even all information about of the last page $E^\infty$. This often makes use of ingenuous arguments only valid in the example or family of examples under consideration. For this reason we add the word **effective** to the above phrase, and by "effectively computing the spectral sequence" we mean *explicitly* determining *all* morphisms on *all* pages of the spectral sequence. Indeed, the definition one finds in standard textbooks like [**Wei94**, Section 5.4] of the spectral sequence associated to a complex of *finite type* carrying a *finite* filtration *is constructive* in the sense that it can be implemented on a computer (see [**Bar09**]). The message of this work is the following:

> If the spectral sequence of a filtered complex is effectively computable, then, with some extra work, the induced filtration on the total (co)homology is effectively computable as well.

By definition, the objects $E^r_{pq}$ of the spectral sequence associated to the filtered complex $C$ are subfactors of the total object $C_{p+q}$ (see Sections 3 and 5). In Section 4 we introduce the notion of a **generalized embedding** to keep track of this information. The central idea of this work is to use the generalized embeddings $E^\infty_{pq} \to C_{p+q}$ to filter the total (co)homology $H_{p+q}(C)$ — also a subfactor of $C_{p+q}$. This is the content of Theorem 5.1.

Effectively computing the induced filtration is not a main stream application of spectral sequences. Very often, especially in topology, the total filtered complex is not completely known, or is of *infinite* type, although the (total) (co)homology is known to be of finite type. But from some page on, the objects of the spectral sequence become *intrinsic* and of *finite type*. Pushing the spectral sequence to convergence and determining the isomorphism type of the low degree total (co)homologies is already highly nontrivial. The reader is referred to [**RS02**] and the impressive program `Kenzo` [**RSS**]. In its current stage, `Kenzo` is able to compute $A_\infty$-structures on cohomology. The goal here is nevertheless of different nature, namely to effectively compute the induced filtration on the *a priori known* (co)homology. The shape of the spectral sequence starting from the *intrinsic* page will also be used to define new numerical invariants of modules and sheaves of modules (cf. Subsection 9.1.5).

The approach favored here makes extensive use of **generalized maps**, a concept motivated in Section 3, introduced in Section 4, and put into action starting from Section 5.

> Generalized maps can be viewed as a *data structure* that allows *reorganizing* many algorithms in homological algebra as *closed formulas*.

Although the whole theoretical content of this work can be done over an abstract abelian category, it is sometimes convenient to be able to refer to elements. The discussion in [**Har77**, p. 203] explains why this can be assumed without loss of generality.

## 2. A generality on subobject lattices

The following situation will be repeatedly encountered in the sequel. Let $C$ be an object in an abelian category, $Z$, $B$, and $A$ subobjects with $B \leq Z$. Then the subobject lattice[3] of $C$ is at most a **degeneration** of the one in Figure 1.

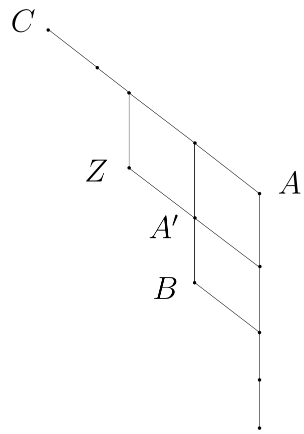



FIGURE 1. A general lattice with subobjects $B \leq Z$ and $A$

This lattice makes no statement about the "size" of $B$ or $Z$ compared to $A$, since, in general, neither $B$ nor $Z$ is in a $\leq$-relation with $A$. The **second**[4] **isomorphism theorem** can be applied ten times within this lattice, two for each of the five parallelograms. The subobject $A$ leads to the **intermediate subobject** $A' := (A+B) \cap Z$ sitting between $B$ and $Z$, which in general neither coincides with $Z$ nor with $B$. Hence, a 2-step filtration $0 \leq A \leq C$ leads to a 2-step filtration $0 \leq A'/B \leq Z/B$.

Arguing in terms of subobject lattices is a manifestation of the isomorphism theorems, all being immediate corollaries of the homomorphism theorem (cf. [**Noe27**]).

## 3. Long exact sequences as spectral sequences

Long exact sequences are in a precise sense a precursor of spectral sequences of filtered complexes. They have the advantage of being a lot easier to comprehend. The core idea around which this work is built can already be illustrated using long exact sequences, which is the aim of this section.

Long exact sequences often occur as the sequence connecting the homologies

$$\cdots \leftarrow H_{n-1}(A) \xleftarrow{\partial_*} H_n(R) \xleftarrow{\nu_*} H_n(C) \xleftarrow{\iota_*} H_n(A) \xleftarrow{\partial_*} H_{n+1}(R) \leftarrow \cdots$$

[3]I learned drawing these pictures from Prof. JOACHIM NEUBÜSER. He made intensive use of subgroup lattices in his courses on finite group theory to visualize arguments and even make proofs.

[4]Here we follow the numbering in EMMY NOETHER's fundamental paper [**Noe27**].

of a *short exact* sequence of complexes $0 \leftarrow R \xleftarrow{\nu} C \xleftarrow{\iota} A \leftarrow 0$. If one views $(A, \partial_A)$ as a subcomplex of $(C, \partial)$, then $(R, \partial_R)$ can be identified with the quotient complex $C/A$. Moreover $\partial_A$ is then $\partial_{|A}$ and $\partial_R$ is boundary operator induced by $\partial$ on the quotient $R$. The natural maps $\partial_*$ appearing in the long exact sequence are the so-called connecting homomorphisms and are, like $\partial_A$ and $\partial_R$, induced by the boundary operator $\partial$ of the total complex $C$.

To see in which sense a long exact sequence is a special case of a spectral sequence of a filtered complex we first recall the definition of a filtered complex.

**Definition 3.1** (Filtered complex)**.** We distinguish between chain and cochain complexes:

(a) A chain of subcomplexes $(F_pC)_{p\in\mathbb{Z}}$ (i.e. $\partial(F_pC_n) \leq F_pC_{n-1}$ for all $n$) of the chain complex $(C_\bullet, \partial)$ is called an **ascending filtration** if $F_{p-1}C \leq F_pC$. The $p$-th **graded part** is the subfactor chain complex defined by $\operatorname{gr}_p C := F_pC/F_{p-1}C$.

(d) A chain of subcomplexes $(F^pC^n)_{p\in\mathbb{Z}}$ (i.e. $\partial(F^pC^n) \leq F^pC^{n+1}$ for all $n$) of the *co*chain complex $(C^\bullet, \partial)$ is called a **descending filtration** if $F^pC \geq F^{p+1}C$. The $p$-th **graded part** is the subfactor cochain complex defined by $\operatorname{gr}^p C := F^pC/F^{p+1}C$.

Like for modules all filtrations of complexes will be **exhaustive** (i.e. $\bigcup_p F_pC = C$), **Hausdorff** (i.e. $\bigcap_p F_pC = 0$), and will have **finite length** $m$ (i.e. the difference between the highest and the lowest stable index is at most $m$). Such filtrations are called $m$-step filtrations in the sequel.

Convention: For the purpose of this work filtrations on chain complexes are automatically ascending whereas on *co*chain complexes descending.

**Remark 3.2.** Before continuing with the previous discussion it is important to note that

(a) The filtration $(F_pC_n)$ of $C_n$ *induces* an ascending filtration on the homology $H_n(C)$. Its $p$-th graded part is denoted by $\operatorname{gr}_p H_n(C)$.

(d) The filtration $(F^pC^n)$ of $C^n$ *induces* a descending filtration on the cohomology $H^n(C)$. Its $p$-th graded parts is denoted by $\operatorname{gr}^p H^n(C)$.

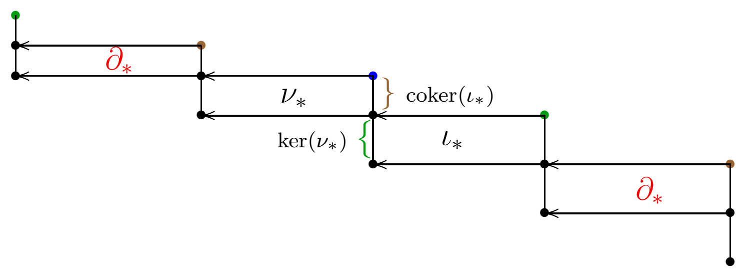
More precisely, $F_pH_n(C)$ is the image of the morphism $H_n(F_pC) \to H_n(C)$.

A short exact sequence of (co)chain complexes $0 \leftarrow R \xleftarrow{\nu} C \xleftarrow{\iota} A \leftarrow 0$ can be viewed as a 2-step filtration $0 \leq A \leq C$ of the complex $C$ with graded parts $A$ and $R$. Following the above convention the filtration is ascending or descending depending on whether $C$ is a chain or cochain complex.

The main idea behind long exact sequences is to relate the homologies of the total chain complex $C$ with the homologies of its graded parts $A$ and $R$. This precisely is also the idea behind spectral sequences of filtered complexes but generalized to $m$-step filtrations, where $m$ may now be larger than 2. Roughly speaking, the spectral sequence of a filtered complex measures how far the graded part $\operatorname{gr}_p H_n(C)$ of the filtered $n$-th homology $H_n(C)$ of the total filtered complex $C$ is away from simply being the homology $H_n(\operatorname{gr}_p C)$ of the $p$-th graded part of $C$. This would for example happen if the filtration $F_pC$ is induced by its own grading[5], i.e. $F_pC = \bigoplus_{p' \leq p} \mathrm{gr}'_p C$, since then the homologies of $C$ will simply be the direct sum of the homologies of the graded parts $\mathrm{gr}_p C$. In general, $\mathrm{gr}_p H_n(C)$ will only be a *subfactor* of $H_n(\mathrm{gr}_p C)$.

Long exact sequences do not have a direct generalization to $m$-step filtrations, $m > 2$. The language of spectral sequences offers in this respect a better alternative. In order to make the transition to the language of spectral sequences notice that the graded parts $\mathrm{coker}(\iota_*)$ and $\ker(\nu_*)$ of the filtered total homology $H_n(C)$ indicated in the diagram below

$$H_{n-1}(A) \xleftarrow{\partial_*} H_n(R) \xleftarrow{\nu_*} H_n(C) \xleftarrow{\iota_*} H_n(A) \xleftarrow{\partial_*} H_{n+1}(R) \tag{1}$$




both have an alternative description in terms of the connecting homomorphisms:

$$\mathrm{coker}(\iota_*) \cong \ker(\partial_*) \qquad \text{and} \qquad \ker(\nu_*) \cong \mathrm{coker}(\partial_*). \tag{2}$$

These natural isomorphisms are nothing but the statement of the homomorphism theorem applied to $\iota_*$ and $\nu_*$.

Below we will give the definition of a spectral sequence and in Section 5 we will recall how to associate a spectral sequence to a filtered complex. But before doing so let us describe in simple words the rough picture, valid for general spectral sequences (even for those not associated to a filtered complex).

A spectral sequence can be viewed as a book with several pages $E^a$, $E^{a+1}$, $E^{a+2}$, ... starting at some integer $a$. Each page contains a double array $E^r_{pq}$ of objects, arranged in an array of complexes. The pattern of arranging the objects in such an array of complexes depends only on the integer $a$ and is fixed by a common convention once and for all. The objects on page $r+1$ are the homologies of the complexes on page $r$. It follows that the object $E^r_{pq}$ on page $r$ are *subfactors* of the objects $E^t_{pq}$ on *all* the previous pages $t < r$.

Now we turn to the morphisms of the complexes. From what we have just been saying we know that at least the source and the target of a morphism on page $r+1$ are completely determined by page $r$. This can be regarded as a sort of restriction on the morphism, and indeed, in the case when zero is the only morphism from the given source to the given target, the morphism then becomes uniquely determined. This happens for example whenever either the source or the target vanishes, but may happen of course in other situations ($\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = 0$). So now it is natural to ask whether page $r$ or any of its previous pages impose further restrictions on the morphisms on page $r+1$, apart from

[5] In the context of long exact sequences this would mean that the short exact sequence of complexes $0 \leftarrow Q \xleftarrow{\nu} C \xleftarrow{\iota} T \leftarrow 0$ splits.

determining their sources and targets. The answer is, in general, no. This will become clear as soon as we construct the spectral sequence associated to a 2-step filtered complex below (or more generally for an $m$-step filtration in Section 5) and understand the nature of data on each page.

Summing up: Taking homology only determines the objects of the complexes on page $r+1$, but not their morphisms. Choosing these morphisms not only completes the $(r+1)$-st page, but again determines the objects on the $(r+2)$-nd page. Iterating this process finally defines a spectral sequence.

Typically, in applications of spectral sequences there exists a natural choice of the morphisms on the successive pages. This is illustrated in the following example, where we associate a spectral sequence to a 2-filtered complex. But first we recall the definition of a spectral sequence.

**Definition 3.3** (Homological spectral sequence)**.** A **homological spectral sequence** (starting at $r_0$) in an abelian category $\mathcal{A}$ consists of

(1) Objects $E^r_{pq} \in \mathcal{A}$, for $p, q, r \in \mathbb{Z}$ and $r \geq r_0 \in \mathbb{Z}$; arranged as a sequence (indexed by $r$) of lattices (indexed by $p, q$);
(2) Morphisms $\partial^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ with $\partial^r \partial^r = 0$, i.e. the sequences of slope $-\frac{r+1}{r}$ in $E^r$ form a chain complex;
(3) Isomorphisms between $E^{r+1}_{pq}$ and the homology $\ker \partial^r_{pq} / \operatorname{im} \partial^r_{p+r,q-r+1}$ of $E^r$ at the spot $(p, q)$.

$E^r$ is called the $r$-th **sheet** (or **page**, or **term**) of the spectral sequence.

Note that $E^{r+1}_{pq}$ is by definition (isomorphic to) a subfactor of $E^r_{pq}$. $p$ is called the **filtration degree** and $q$ the **complementary degree**. The sum $n = p + q$ is called the **total degree**. A morphism with source of total degree $n$, i.e. on the $n$-th diagonal, has target of degree $n-1$, i.e. on the $(n-1)$-st diagonal. So the total degree is *decreased* by one.

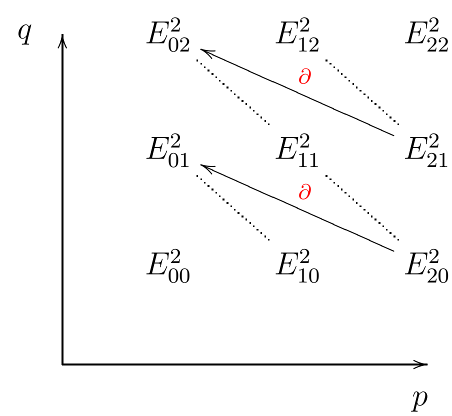



FIGURE 2. $E^2$

**Definition 3.4** (Cohomological spectral sequence)**.** A **cohomological spectral sequence** (starting at $r_0$) in an abelian category $\mathcal{A}$ consists of

(1) Objects $E_r^{pq} \in \mathcal{A}$, for $p, q, r \in \mathbb{Z}$ and $r \geq r_0 \in \mathbb{Z}$; arranged as a sequence (indexed by $r$) of lattices (indexes by $p, q$);
(2) Morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ with $d_r d_r = 0$, i.e. the sequences of slope $-\frac{r+1}{r}$ in $E_r$ form a cochain complex;
(3) Isomorphisms between $E_{r+1}^{pq}$ and the cohomology of $E_r$ at the spot $(p, q)$.

$E_r$ is called the $r$-th **sheet** of the spectral sequence.

Here the total degree $n = p + q$ is *increased* by one. Reflecting a cohomological spectral sequence at the origin $(p, q) = (0, 0)$, for example, defines a homological one $E^r_{pq} = E_r^{-p,-q}$, and vice versa. For more details and terminology (**boundedness**, **convergence**, **fiber terms**, **base terms**, **edge homomorphisms**, **collapsing**, $E^\infty$ **term**, **regularity**) see [**Wei94**, Section 5.2].

Part of the data we have in the context of long exact sequences can be put together to construct a spectral sequence with three pages $E^0$, $E^1$, and $E^2$:

$$E^0_{pq}: \quad \begin{tikzcd} A_n \arrow[dr, dotted, no head] & R_{n+1} \\ A_{n-1} \arrow[dr, dotted, no head] & R_n \\ A_{n-2} & R_{n-1} \end{tikzcd} \qquad \overset{\text{add the}}{\underset{\text{arrows}}{\rightsquigarrow}} \qquad E^0_{pq}: \quad \begin{tikzcd} A_n \arrow[d, "\partial_A"] \arrow[dr, dotted, no head] & R_{n+1} \arrow[d, "\partial_R"] \\ A_{n-1} \arrow[d, "\partial_A"] \arrow[dr, dotted, no head] & R_n \arrow[d, "\partial_R"] \\ A_{n-2} & R_{n-1} \end{tikzcd}$$

take $\rightsquigarrow$ homology

$$E^1_{pq}: \quad \begin{tikzcd} H_n(A) \arrow[dr, dotted, no head] & H_{n+1}(R) \\ H_{n-1}(A) \arrow[dr, dotted, no head] & H_n(R) \\ H_{n-2}(A) & H_{n-1}(R) \end{tikzcd} \qquad \overset{\text{add the}}{\underset{\text{arrows}}{\rightsquigarrow}} \qquad E^1_{pq}: \quad \begin{tikzcd} H_n(A) \arrow[dr, dotted, no head] & H_{n+1}(R) \arrow[l, "\partial_*"'] \\ H_{n-1}(A) \arrow[dr, dotted, no head] & H_n(R) \arrow[l, "\partial_*"'] \\ H_{n-2}(A) & H_{n-1}(R) \arrow[l, "\partial_*"'] \end{tikzcd}$$

take $\rightsquigarrow$ homology

$$\begin{tikzcd}
E^2_{pq}: & \operatorname{coker}(\partial_*) \arrow[dr, dash, dotted] & \ker(\partial_*) & & E^2_{pq}: & \operatorname{coker}(\partial_*) \arrow[dr, dash, dotted] & \ker(\partial_*) \\
& \operatorname{coker}(\partial_*) \arrow[dr, dash, dotted] & \ker(\partial_*) \arrow[rr, phantom, "\overset{\text{no arrows}}{\underset{\text{to add}}{\rightsquigarrow}}"] & & & \operatorname{coker}(\partial_*) \arrow[dr, dash, dotted] & \ker(\partial_*) \\
& \operatorname{coker}(\partial_*) & \ker(\partial_*) & & & \operatorname{coker}(\partial_*) & \ker(\partial_*)
\end{tikzcd}$$

with $p, q \in \mathbb{Z}$, $n = p + q$. Taking the two columns over $p = 0$ and $p = 1$, for example, is equivalent to setting $F_{-1}C := 0$, $F_0C := A$, and $F_1C := C$.

Several remarks are in order. First note that all the arrows in the above spectral sequence are induced by $\partial$, the boundary operator of the total complex $C$. Since $\partial$ respects the filtration, i.e. $\partial(F_pC) \leq F_pC$, the induced map $\bar{\partial} : F_pC \to C/F_pC$ vanishes. So respecting the filtration means that $\partial$ cannot carry things up in the filtration. But since $\partial$ does not necessarily respect the grading induced by the filtration it may very well carry things down one or more levels. Now we can interpret the pages: $E^0$ consists of the graded parts $\operatorname{gr}_p C$ with boundary operators $\partial_A$ and $\partial_Q$ chopping off all what $\partial$ carries down in the filtration. $E^1$ describes what $\partial$ carries down exactly one level. This interpretation of the connecting homomorphisms $\partial_*$ puts them on the same conceptual level as $\partial_A$ and $\partial_Q$. Finally, $E^2$ describes what $\partial$ carries exactly two levels down, but since a 2-step filtration has two levels it should now be clear why $E^2$ does not have arrows.

Second, as we have seen in (2) using the homomorphism theorem, the objects of the last page $E^2$ can be naturally identified with the graded parts $\operatorname{gr}_p H_n(C)$ of the filtered total homology $H_n(C)$. And since the objects on each page are subfactors of the objects on the previous pages one can view the above spectral sequence as a process successively approximating the graded parts $\operatorname{gr}_p H_n(C)$ of the filtered total homology $H_n(C)$:

$$(A_n, R_n) \rightsquigarrow (H_n(A), H_n(R)) \rightsquigarrow (\operatorname{coker}(\partial_*), \ker(\partial_*)).$$

The approximation is achieved by successively taking deeper inter-level interaction into account.

Finally one can ask if the spectral sequence above captured all the information in the long exact sequence. The answer is *no*. The long exact sequence additionally contains the short exact sequence

$$0 \leftarrow \ker(\partial_*) \xleftarrow{\nu_*} H_n(C) \xleftarrow{\iota_*} \operatorname{coker}(\partial_*) \leftarrow 0, \tag{3}$$

explicitly describing the total homology $H_n(C)$ as an extension of its graded parts $\operatorname{coker}(\partial_*)$ and $\ker(\partial_*)$.

Looking to what happens inside the subobject lattice of $C_n$ during the approximation process will help understanding how to remedy this defect.

Figure 3 shows the $n$-th object $C_n$ in the chain complex together with the subobjects that define the different homologies: $H_n(R) := Z_n(R)/B_n(R)$, $H_n(A) := Z_n(A)/B_n(A)$, and $H_n(C) := Z_n(C)/B_n(C)$. Here we replaced $Z_n(R)$ and $B_n(R)$ by their full preimages in $C_n$ under the canonical epimorphism $C_n \xrightarrow{\nu} R_n := C_n/A_n$.

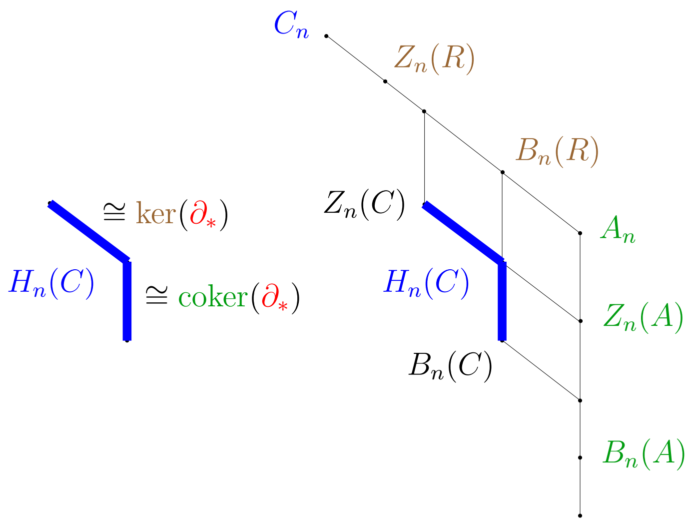
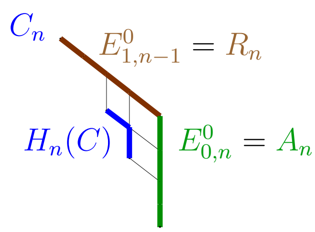
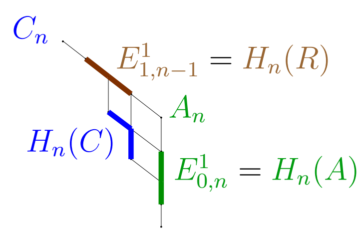



FIGURE 3. The 2-step filtration $0 \leq A \leq C$ and the induced 2-step filtration on $H_*(C)$




$\rightsquigarrow$




$\rightsquigarrow$

FIGURE 4. $E^0$

FIGURE 5. $E^1$

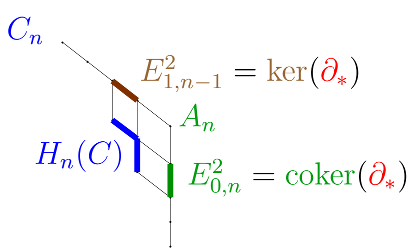



FIGURE 6. $E^2 = E^\infty$

The approximation process of the graded parts of $H_n(C)$

Figures 4-6 show how the graded parts of $H_n(C)$ get successively approximated by the objects in the spectral sequence $E^r_{pq}$, naturally identified with certain subfactors of $C_n$ for $n = p + q$. Figure 6 proves that the second isomorphism theorem provides *canonical* isomorphisms between the graded parts of the total homology $H_n(C)$ and the objects $E^\infty_{1,n-1} = E^2_{1,n-1}$ and $E^\infty_{0,n} = E^2_{0,n}$ of the stable sheet. And modulo these natural isomorphisms Figure 6 further suggests that knowing how to identify $E^\infty_{1,n-1}$ and $E^\infty_{0,n}$ with the indicated subfactors of $C_n$ will suffice to explicitly construct the extension (3) in the form

$$0 \leftarrow E^{\infty}_{1,n-1} \leftarrow H_n(C) \leftarrow E^{\infty}_{0,n} \leftarrow 0. \tag{4}$$

But since we cannot use maps to identify objects with subfactors of other objects we are lead to introduce the notion of **generalized maps** in the next Section. Roughly speaking, this notion enables us to interpret the pairs of horizontal arrows in Figure 7 as **generalized embeddings**.

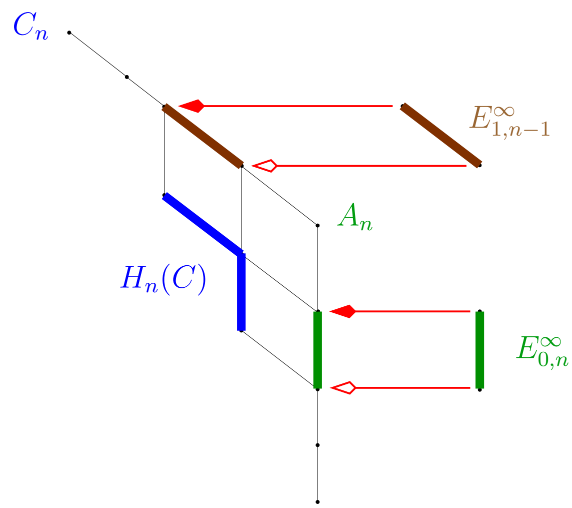



FIGURE 7. The generalized embeddings

## 4. Generalized maps

A morphism between two objects (modules, complexes, ...) induces a map between their lattice of subobjects, and the **homomorphism theorem** implies that this map gives rise to a bijective correspondence between the subobjects of the target lying in the image and those subobjects of the source containing the kernel. This motivates the visualization in Figure 8 of a morphism $T \xleftarrow{\varphi} S$ with source $S$ and target $T$. The homomorphism theorem states that the morphism $\varphi$, indicated by the horizontal pair of arrows in Figure 8, maps $S/\ker(\varphi)$ onto the *subobject* $\mathrm{im}(\varphi)$ in a structure-preserving way. In this sense, the exact ladder of morphisms in (1) visualizes part of the long exact homology sequence.

The simplest motivation for the notion of a generalized morphism $T \xleftarrow{\psi} S$ is the desire to give sense to the picture in Figure 9 "mapping" a quotient of $S$ onto a *subfactor* of $T$.

**Definition 4.1** (Generalized morphism)**.** Let $S$ and $T$ be two objects in an abelian category (of modules over some ring). A **generalized morphism** $\psi$ with source $S$ and target $T$ is a pair of morphisms $(\bar{\psi}, \imath)$, where $\imath$ is a morphism from some third object $F$ to $T$ and $\bar{\psi}$ is a morphism from $S$ to $\operatorname{coker} \imath = T/\operatorname{im}(\imath)$. We call $\bar{\psi}$ the morphism **associated** to $\psi$ and $\imath$ the **morphism aid** of $\psi$ and denote it by $\operatorname{Aid}\psi$. Further we call $L := \operatorname{im}\imath \leq T$ the **morphism aid subobject**. Two generalized morphisms $(\bar{\psi}, \imath)$ and $(\bar{\varphi}, \jmath)$ with ($\operatorname{im}\imath = \operatorname{im}\jmath$ and) $\bar{\psi} = \bar{\varphi}$ will be identified.

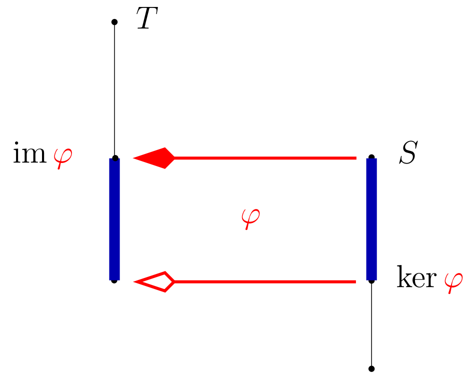



FIGURE 8. The homomorphism theorem

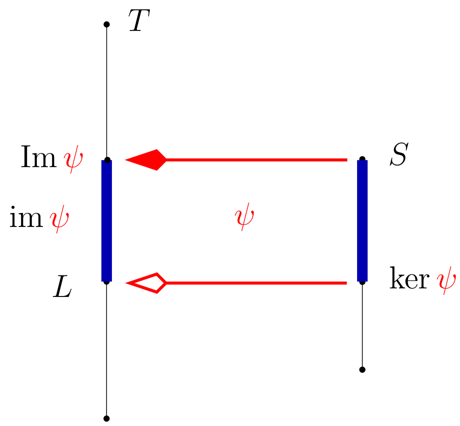



FIGURE 9. A generalized morphism

Philosophically speaking, this definition frees one from the "conservative" standpoint of viewing $\psi$ as morphism to the quotient $T/\operatorname{im}\imath$. Instead it allows one to view $\psi$ as a "morphism" to the full object $T$ by directly incorporating $\imath$ in the very definition of $\psi$. The intuition behind the notion "morphism aid" (resp. "morphism aid subobject") is that $\imath$ (resp. $L = \operatorname{im}\imath$) *aids* $\psi$ to become a (well-defined) morphism. Figure 10 visualizes the generalized morphism $\psi$ as a pair $(\bar{\psi}, \imath)$.

Note that replacing $\imath$ by a morphism with the same image does not alter the generalized morphism. We will therefore often write $(\bar{\psi}, L)$ for the generalized morphism $(\bar{\psi}, \imath)$, where $\imath$ is any morphism with $\operatorname{im}\imath = L \leq T$. The most natural choice would be the embedding $\imath : L \to T$. Figure 9 visualizes the generalized morphism $\psi$ as a pair $(\bar{\psi}, L)$. It also reflects the idea behind the definition more than the "expanded" Figure 10 does.

If $L = \operatorname{im}\imath$ vanishes, then $\psi$ is nothing but the (ordinary) morphism $\bar{\psi}$. Conversely, any morphism can be viewed as a generalized morphism with trivial morphism aid subobject $L = 0$.

**Definition 4.2** (Terminology for generalized morphisms)**.** Let $\psi = (\bar{\psi}, \imath) : S \to T$ be a generalized morphism. Define the **kernel** $\ker(\psi) := \ker\bar{\psi}$, the kernel of the associated

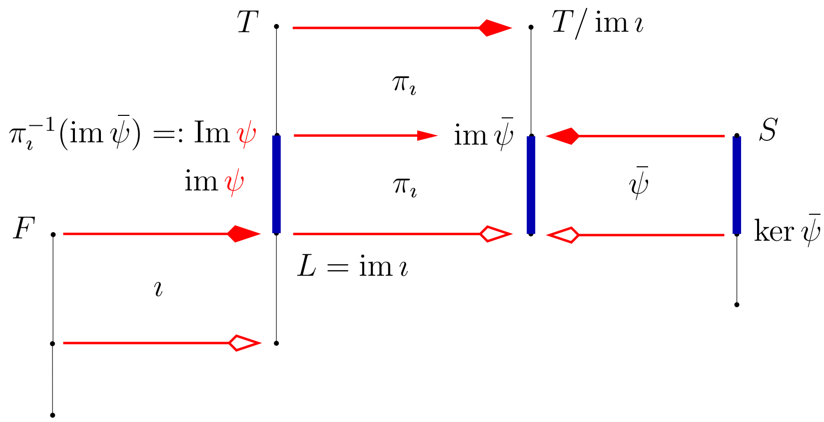



FIGURE 10. The morphism aid $\imath$ and the associated morphism $\bar\psi$

map. If $\pi_\imath$ denotes the natural epimorphism $T \to T/\operatorname{im}\imath$, then define the **combined image** $\operatorname{Im}\psi$ to be the *submodule* $\pi_\imath^{-1}(\operatorname{im}\bar\psi)$ of $T$. In general it differs from the **image** $\operatorname{im}\psi$ which is defined as the *subfactor* $\operatorname{Im}\psi/\operatorname{im}\imath$ of $T$ (cf. Figure 10). We call $\psi$ a **generalized monomorphism** (resp. **generalized epimorphism**, **generalized isomorphism**) if the associated map $\bar\psi$ is a monomorphism (resp. epimorphism, isomorphism).

Sometimes we use the terminology **generalized map** instead of generalized morphism and **generalized embedding** instead of generalized monomorphism, especially when the abelian category is a category of modules (or complexes of modules, etc.).

As a first application of the notion of generalized embeddings we state the following definition, which is central for this work.

**Definition 4.3** (Filtration system)**.** Let $\mathfrak{I} = (p_0, \ldots, p_{m-1})$ be a finite interval in $\mathbb{Z}$, i.e. $p_{i+1} = p_i + 1$.
A finite sequence of generalized embeddings $\psi_p = (\bar\psi_p, L_p)$, $p \in \mathfrak{I}$ with common target $M$ is called an **ascending $m$-filtration system** of $M$ if

(1) $\psi_{p_0}$ is an ordinary monomorphism, i.e. $L_{p_0}$ vanishes;
(2) $L_p = \operatorname{Im}\psi_{p-1}$, for $p = p_1, \ldots, p_{m-1}$;
(3) $\psi_{p_{m-1}}$ is a generalized isomorphism, i.e. $\operatorname{Im}\psi_{p_{m-1}} = M$.

A finite sequence of generalized embeddings $\psi^p = (\bar\psi^p, L^p)$, $p \in \mathfrak{I}$ with common target $M$ is called a **descending $m$-filtration system** of $M$ if

(1) $\psi^{p_0}$ is a generalized isomorphism, i.e. $\operatorname{Im}\psi^{p_0} = M$;
(2) $L^p = \operatorname{Im}\psi^{p+1}$, for $p = p_0, \ldots, p_{m-2}$;
(3) $\psi^{p_{m-1}}$ is an ordinary monomorphism, i.e. $L^{p_{m-1}}$ vanishes.

We say $(\psi_p)$ **computes** a given filtration $(F_pM)$ if $\operatorname{Im}\psi_p = F_pM$ for all $p$.

Now we come to the definition of the basic operations for generalized morphisms. Two generalized maps $\psi = (\bar\psi, \imath)$ and $\varphi = (\bar\varphi, \jmath)$ are summable only if $\operatorname{im}\imath = \operatorname{im}\jmath$ and we set $\psi \pm \varphi := (\bar\psi \pm \bar\varphi, \imath)$.

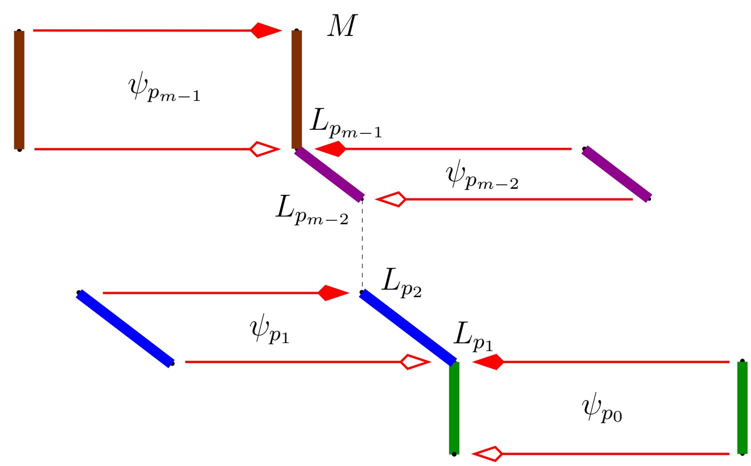



FIGURE 11. An ascending $m$-filtration system

The following notational convention will prove useful: It will often happen that one wants to alter a generalized morphism $\psi = (\bar{\psi}, L_\psi)$ with target $T$ by replacing $L_\psi$ with a larger subobject $L$, i.e. a subobject $L \leq T$ containing $L_\psi$. We will sloppily write $\widetilde{\psi} = (\bar{\psi}, L)$, where $\bar{\psi}$ now stands for the composition of $\bar{\psi}$ with the natural epimorphism $T/L_\psi \to T/L$. We will say that $\psi$ was **coarsened** to $\widetilde{\psi}$ to refer to the passage from $\psi = (\bar{\psi}, L_\psi)$ to $\widetilde{\psi} = (\bar{\psi}, L)$ with $L_\psi \leq L \leq T$. As Figure 12 shows, coarsening $\psi$ might very well enlarge its combined image $\operatorname{Im}\psi$. The word "coarse" refers to the fact that the image $\operatorname{im}\widetilde{\psi}$ is naturally isomorphic to a *quotient* of $\operatorname{im}\psi$, and Figure 12 shows that this natural isomorphism is given by the second isomorphism theorem. We say that the coarsening $\widetilde{\psi} = (\bar{\psi}, L)$ of $\psi = (\bar{\psi}, L_\psi)$ is **effective**, if $\operatorname{Im}\psi \cap L = L_\psi$. Figure 12 shows that in this case the images $\operatorname{im}\psi$ and $\operatorname{im}\widetilde{\psi}$ are naturally *isomorphic*.

For the composition $\psi \circ \varphi$ of $S_\varphi \xrightarrow{\varphi} T_\varphi = S_\psi \xrightarrow{\psi} T_\psi$ follow the filled area in Figure 13 from left to right.

Formally, first coarsen $\varphi = (\bar{\varphi}, \jmath) \to \widetilde{\varphi} = (\bar{\varphi}, K)$, where

$$K := \operatorname{im}\jmath + \ker\psi \leq T_\varphi.$$

Then coarsen $\psi = (\bar{\psi}, \imath) \to \widetilde{\psi} = (\bar{\psi}, L)$, where

$$L := \pi_\imath^{-1}(\operatorname{im}(\bar{\psi} \circ \jmath)) = \pi_\imath^{-1}(\bar{\psi}(K)) \leq T_\psi$$

and $\pi_\imath$ as above. Now set

$$\psi \circ \varphi := (\bar{\psi} \circ \bar{\varphi}, L).$$

Note that $\ker\psi \circ \varphi = \ker\widetilde{\varphi}$.

Finally we define the division $\beta^{-1} \circ \gamma$ of two generalized maps $S_\gamma \xrightarrow{\gamma} T \xleftarrow{\beta} S_\beta$ under the conditions of the next definition.

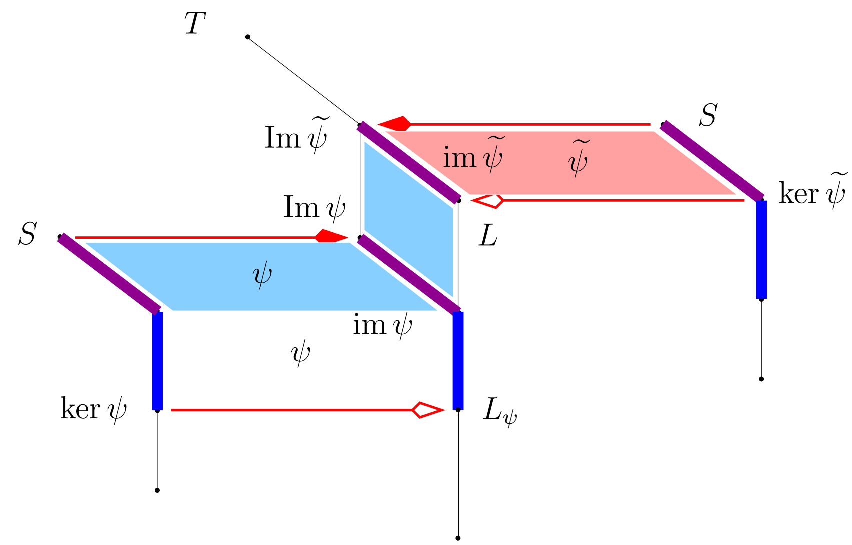



FIGURE 12. Coarsening the generalized map $\psi = (\bar{\psi}, K)$ to $\widetilde{\psi} = (\bar{\psi}, L)$

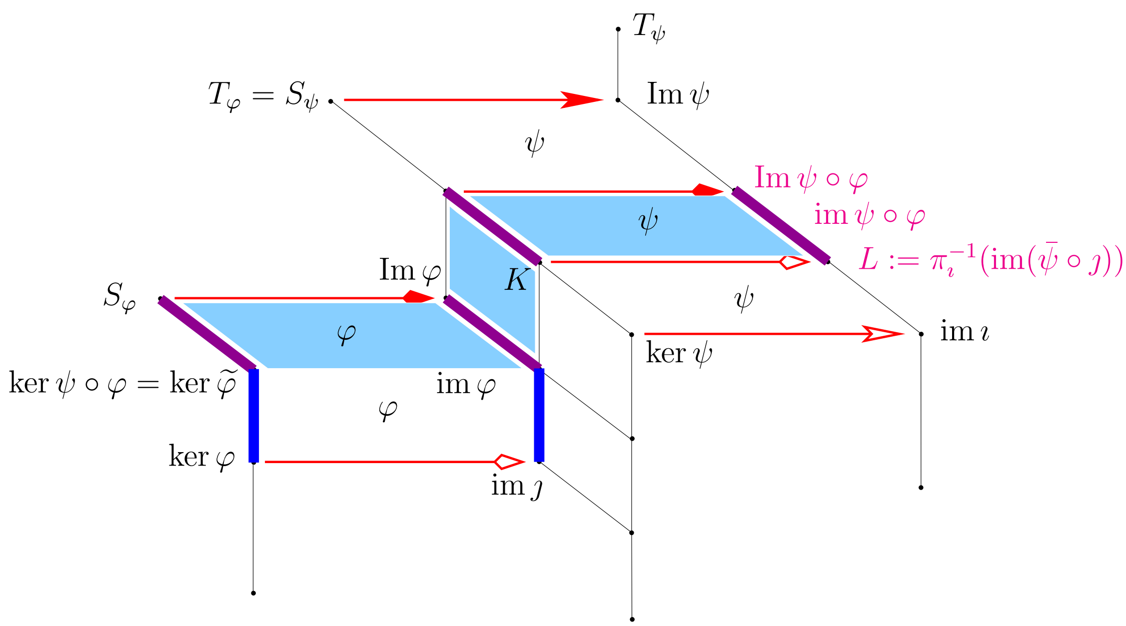



FIGURE 13. The composition $\psi \circ \varphi$

**Definition 4.4** (The lifting condition)**.** Let $\gamma = (\bar{\gamma}, L_\gamma)$ and $\beta = (\bar{\beta}, L_\beta)$ be two generalized morphisms with the same target $N$.

$$\begin{tikzcd}
M' \arrow[dr, "\gamma"] & \\
N' \arrow[r, "\beta"'] & N.
\end{tikzcd}$$

Consider the **common coarsening** of the generalized maps $\beta$ and $\gamma$, i.e. the generalized maps $\widetilde{\beta} := (\bar{\beta}, L)$ and $\widetilde{\gamma} := (\bar{\gamma}, L)$, where $L = L_\gamma + L_\beta \leq N$. We say $\beta$ **lifts** $\gamma$ (or **divides** $\gamma$) if the following two conditions are satisfied:

**(im)** The combined image of $\widetilde{\beta}$ contains the combined image of $\widetilde{\gamma}$:

$$\operatorname{Im}\widetilde{\gamma} \leq \operatorname{Im}\widetilde{\beta}.$$

**(eff)** The coarsening $\gamma \to \widetilde{\gamma}$ is effective, i.e. $\operatorname{Im}\gamma \cap L = L_\gamma$.

We will refer to $\widetilde{\gamma}$ as **the effective coarsening of** $\gamma$ **with respect to** $\beta$. The following lemma justifies this definition. Both the definition and the lemma are visualized in Figure 14. To state the lemma one last notion is needed: Define two generalized morphisms $\psi = (\bar{\psi}, L_\psi)$ and $\varphi = (\bar{\varphi}, L_\varphi)$ to be **equal up to effective common coarsening** or **quasi-equal** if their common coarsenings $\widetilde{\psi} := (\overline{\psi}, L)$ and $\widetilde{\varphi} := (\overline{\varphi}, L)$ coincide *and* are *both* effective. We write $\psi \triangleq \varphi$.

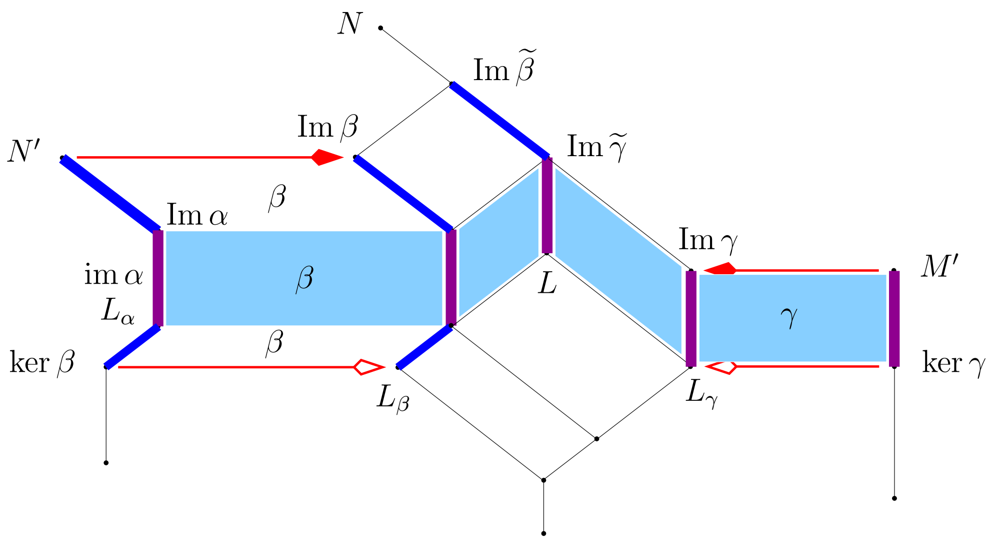



FIGURE 14. The lifting condition and the lifting lemma

**Lemma 4.5** (The lifting lemma). *Let $\gamma = (\bar{\gamma}, L_\gamma)$ and $\beta = (\bar{\beta}, L_\beta)$ be two generalized morphisms with the same target $N$. Suppose that $\beta$ lifts $\gamma$. Then there exists a generalized morphism $\alpha : M' \to N'$ with $\beta \circ \alpha \triangleq \gamma$,*

$$\begin{tikzcd} M' \arrow[d, "\alpha"] \arrow[dr, "\gamma"] & \\ N' \arrow[r, "\beta"'] & N. \end{tikzcd}$$

*i.e. $\beta \circ \alpha$ is equal to $\gamma$ up to effective common coarsening. $\alpha$ is called* ***a lift*** *of $\gamma$ along $\beta$. Further let $\widetilde{\gamma} := (\bar{\gamma}, L_{\widetilde{\gamma}})$ be the effective coarsening of $\gamma$ with respect to $\beta$, i.e. $L_{\widetilde{\gamma}} = L = L_\gamma + L_\beta$. Then there exists a* unique *lift $\alpha = (\bar{\alpha}, L_\alpha)$ satisfying*

(a) $\operatorname{Im}\alpha = \bar{\beta}^{-1}(\operatorname{Im}\widetilde{\gamma})$ *and*
(b) $L_\alpha = \bar{\beta}^{-1}(L_{\widetilde{\gamma}})$.

*This $\alpha$ is called **the lift** of $\gamma$ along $\beta$, or **the quotient** of $\gamma$ by $\beta$ and is denoted by $\beta^{-1} \circ \gamma$ or by $\gamma/\beta$.*

PROOF. The subobject lattice(s) in Figure 14 describes the most general setup imposed by conditions (im) and (eff), in the sense that all other subobject lattices of configurations satisfying these two conditions are at most degenerations of the one in Figure 14. Now to construct the unique $\alpha$ simply follow the filled area from right to left. $\square$

The reader may have already noticed that the choice of the symbol $\triangleq$ for quasi-equality was motivated by Figure 14, with $L$ at the tip of the pyramid. The proof makes it clear that the lifting lemma is yet another incarnation of the homomorphism theorem.

**Remark 4.6** (Effective computability)**.** Note that the lift $\alpha = (\bar{\alpha}, L_\alpha)$ sees from $N'$ only its subfactor $N'/L_\alpha$. Replacing $N'$ by its subfactor $N'/L_\alpha$ turns $\beta$ into a generalized embedding, which we again denote by $\beta$. Now $\gamma$ and *this* $\beta$ have effective common coarsenings $\widetilde{\gamma} = (\bar{\gamma}, L)$ and $\widetilde{\beta} = (\bar{\beta}, L)$, which see from $N$ only $N/L$, where $L = L_\gamma + L_\beta$. And modulo $L$ the generalized morphism $\widetilde{\gamma}$ becomes a morphism and the generalized embedding $\widetilde{\beta}$ becomes an (ordinary) embedding. So from the point of view of effective computations the setup can be reduced to the following situation: $\gamma : M' \to N$ is a morphism and $\beta : N' \to N$ is a *monomorphism*. When $M'$, $N'$, and $N$ are finitely presented modules over a **computable ring** (cf. Def. A.1) it was shown in [**BR08**, Subsection 3.1.1] that in this case the unique morphism $\alpha : M' \to N$ is **effectively** computable.

With the notion of a generalized embedding at our disposal we can finally give the horizontal arrows in Figure 7 a meaning. Now consider the three generalized embeddings $\iota : H_n(C) \to C_n$, $\iota_0 : E^\infty_{0,n} \to C_n$, and $\iota_1 : E^\infty_{1,n-1} \to C_n$ in Figure 15. $\iota_p$ is called the **total embedding** of $E^\infty_{p,n-p}$.

**Corollary 4.7.** *The generalized embedding $\iota$ in Figure 15 lifts both total embeddings $\iota_0$ and $\iota_1$. Thus the two lifts $\epsilon_0 := \iota_0/\iota$ and $\epsilon_1 := \iota_1/\iota$ are generalized embeddings that form a filtration system of $H_n(C)$, visualized in Figure 16. More precisely, $\epsilon_0$ is an (ordinary) embedding and $\epsilon_1$ is a generalized isomorphism.*

PROOF. There are two obvious degenerations of the subobject lattice(s) in Figure 14, both leading to a sublattice of the lattice in Figure 15, one for the pair $(\beta, \gamma) = (\iota, \iota_0)$ and the other for $(\beta, \gamma) = (\iota, \iota_1)$. In other words: Following the two filled areas from right to left constructs $\epsilon_0 := \iota^{-1} \circ \iota_0$ and $\epsilon_1 := \iota^{-1} \circ \iota_1$. $\square$

**Corollary 4.8** (Generalized inverse)**.** *Let $\psi : S \to T$ be a generalized epimorphism. Then there exists a* unique *generalized epimorphism $\psi^{-1} : T \to S$, such that $\psi^{-1} \circ \psi = (\mathrm{id}_S, \ker \psi)$ and $\psi \circ \psi^{-1} = (\mathrm{id}_T, \operatorname{Aid} \psi)$. $\psi^{-1}$ is called the **generalized inverse** of $\psi$. In particular, if $\psi$ is an (ordinary) epimorphism, then $\psi^{-1}$ is a generalized isomorphism, and vice versa.*

PROOF. Since $\psi$ lifts $\mathrm{id}_T$ define $\psi^{-1} := \mathrm{id}_T/\psi$. $\square$

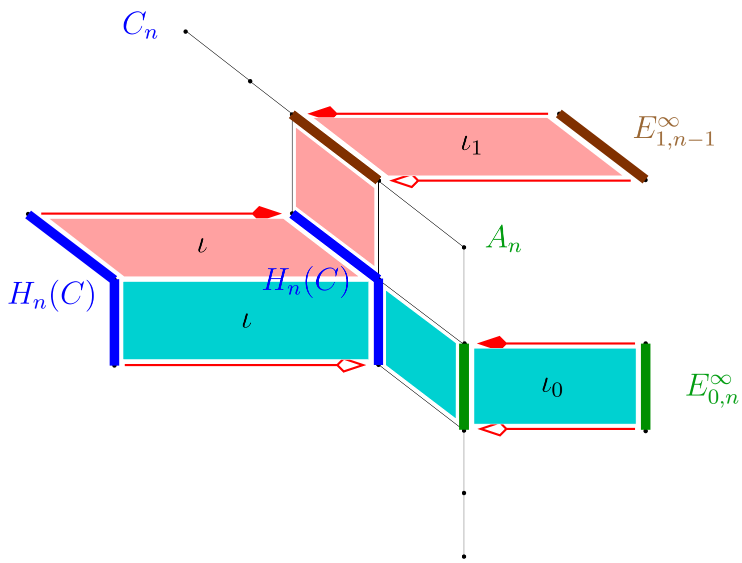



FIGURE 15. $\iota$ lifts $\iota_0$ and $\iota_1$

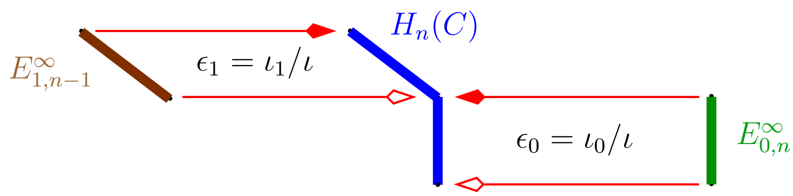



FIGURE 16. The filtration of $H_n(C)$ given by the 2-filtration system $\epsilon_0$, $\epsilon_1$

Rephrasing short exact sequences (also called 1-extensions) in terms of 2-filtration systems is now an easy application of this corollary. In particular, the information in the short exact sequence (4) is fully captured by the 2-filtration system in Figure 16. This is last step of remedying the defect mentioned while introducing the short exact sequence (3) in Section 3.

## 5. Spectral sequences of filtered complexes

Everything substantial already happened in Sections 3 and 4. Here we only show how the ideas already developed for 2-filtrations and their 2-step spectral sequences easily generalize to $m$-filtrations and their $m$-step spectral sequences.

We start by recalling the construction of the **spectral sequence associated to a filtered complex**. The exposition till Theorem 5.1 closely follows [**Wei94**, Section 5.4]. We also remain loyal to our use of subobject lattices as they are able to sum up a considerable amount of relations in one picture.

Consider a chain complex $C$ with (an ascending) filtration $F_pC$. The complementary degree $q$ and the total degree $n$ are dropped for better readability. Define the natural projection $F_pC \to F_pC/F_{p-1}C =: E^0_p$. It is elementary to check that the **subobjects of**

**$r$-approximate cycles**

$$A_p^r := \ker(F_pC \to F_pC/F_{p-r}C) = \{c \in F_pC \mid \partial c \in F_{p-r}C\}$$

satisfy the relations of Figure 17, with $Z_p^r := A_p^r + F_{p-1}C$, $B_p^r := \partial A_{p+(r-1)}^{r-1} + F_{p-1}C$, and $E_p^r := Z_p^r/B_p^r$. These definitions deviate a bit from those in [**Wei94**, Section 5.4]. Here $Z_p^r$ and $B_p^r$ sit between $F_pC$ and $F_{p-1}C$. His $Z_p^r$ and $B_p^r$ are the projections under $\eta_p$ onto $E_p^0 := F_pC/F_{p-1}C$ of the ones here, and hence sit in the objects of the 0-th sheet $E_p^0$. The subobject lattice in Figure 17 should by now be considered an old friend as it is ubiquitous throughout all our arguments.

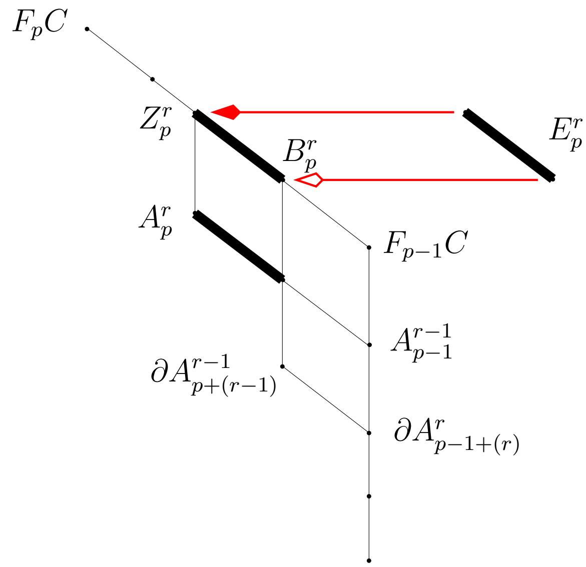



Figure 17. The fundamental subobject lattice

Setting $Z_p^\infty := \cap_{r=0}^\infty Z_p^r$ and $B_p^\infty := \cup_{r=0}^\infty B_p^r$ completes the **tower** of subobjects

$$F_{p-1}C = B_p^0 \leq B_p^1 \leq \cdots \leq B_p^r \leq \cdots \leq B_p^\infty \leq Z_p^\infty \leq \cdots \leq Z_p^r \leq \cdots \leq Z_p^1 \leq Z_p^0 = F_pC$$

between $F_{p-1}C$ and $F_pC$.

From Figure 17 it is immediate that

$$E_p^r := \frac{Z_p^r}{B_p^r} \cong \frac{A_p^r}{\partial A_{p+(r-1)}^{r-1} + A_{p-1}^{r-1}}.$$

It is now routine to verify that the total boundary operator $\partial$ induces morphisms

$$\partial_p^r : E_p^r \to E_{p-r}^r.$$

And as mentioned in Section 3 these morphisms decrease the filtration degree by $r$. They complete the definition of the $r$-th sheet.

From the point of view of effective computations the above definition of $\partial_p^r$ *is constructive*, as long as all involved objects are of *finite type*. In fact, it can easily be turned into an algorithm using generalized maps. But since the filtered complexes relevant to our applications are total complexes of bicomplexes, the description of this algorithm is deferred to Section 6, where the bicomplex structure will be exploited.

To see that $(E^r)$ indeed defines a spectral sequence it remains to show the taking homology in $E^r$ reproduces the objects of $E^{r+1}$ up to (natural) isomorphisms. For this purpose one uses the statements encoded in Figure 17 to deduce that

(a) $Z_p^r / Z_p^{r+1} \cong B_{p-r}^{r+1} / B_{p-r}^r$,
(b) $\ker \partial_p^r \cong Z_p^{r+1} / B_p^r$,
(c) $\operatorname{im} \partial_{p+r}^r \cong B_p^{r+1} / B_p^r$, and finally
(d) $E_p^{r+1} \cong \ker \partial_p^r / \operatorname{im} \partial_{p+r}^r$.

(c) follows from (a) and (b) since they state that $\partial_p^r$ decomposes as

$$E_p^r := Z_p^r / B_p^r \xrightarrow{(b)} Z_p^r / Z_p^{r+1} \xrightarrow{(a)} B_{p-r}^{r+1} / B_{p-r}^r \hookrightarrow Z_{p-r}^r / B_{p-r}^r =: E_{p-r}^r,$$

showing that $\operatorname{im} \partial_p^r \cong B_p^{r+1} / B_p^r$. Now replace $p$ by $p + r$. (d) is the first isomorphism theorem applied to $E_p^{r+1} := Z_p^{r+1} / B_p^{r+1}$ using (b) and (c). For (a) and (b) see [**Wei94**, Lemma 5.4.7 and the subsequent discussion].

Before stating the main theorem we make some remarks about convergence. Recall that all our filtrations are assumed finite of length $m$. This means that $E^m$ runs out of arrows and thus stabilizes, i.e. $E^m = E^{m+1} = \cdots$. We already saw this for $m = 2$ in Section 3. As customary, the stable sheet is denoted by $E^\infty$. The stable form of Figure 17 is Figure 18, where $A_p^\infty := \cup_{r=0}^\infty A_p^r$ and $A_{p+\infty}^\infty := \cup_{r=0}^\infty A_{p+r}^r$.

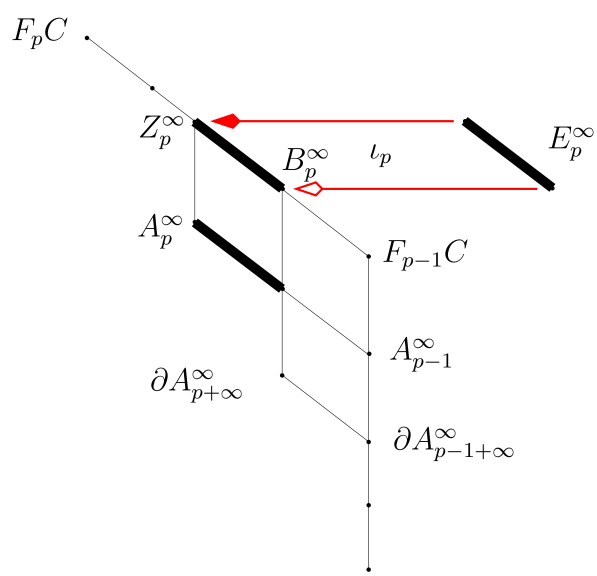



FIGURE 18. The stable fundamental subobject lattice

The identities

$$A_p^\infty = \ker \partial_{|F_pC} = \{c \in F_pC \mid \partial c = 0\} \tag{5}$$

and

$$\partial A^\infty_{p+\infty} = \operatorname{im} \partial_{|F_pC} = \partial C \cap F_pC \tag{6}$$

are direct consequences of the respective definitions.

**Theorem 5.1** (Beyond $E^\infty$)**.** *Let $C$ be a chain complex with an ascending $m$-step filtration. The generalized embedding $\iota : H(C) \to C$ divides all generalized embeddings $\iota_p : E^\infty_p \to C$, called the* ***total embedding*** *of $E^\infty_p$. The quotients $\epsilon_p := \iota_p/\iota$ form an $m$-filtration system which computes the induced filtration on $H(C)$.*

Proof. We only need to verify the two lifting conditions for the pairs $(\iota, \iota_p)$. Everything else is immediate. For the morphism aid subobjects of $\iota_p$ and $\iota$ we have

$$L_{\iota_p} = \partial A^\infty_{p+\infty} + F_{p-1}C$$

(see Figure 18) and

$$L_\iota = \partial C.$$

Define

$$L := L_{\iota_p} + L_\iota = (\partial A^\infty_{p+\infty} + F_{p-1}C) + \partial C = \partial C + F_{p-1}C.$$

**Condition (im)**: Since $\operatorname{Im} \iota_p = A^\infty_p + F_{p-1}C$ and $\operatorname{Im} \iota = \ker \partial$ we obtain

$$\begin{aligned} \operatorname{Im} \widetilde{\iota_p} \leq \operatorname{Im} \widetilde{\iota} \quad &\iff \quad (A^\infty_p + F_{p-1}C) + L \leq \ker \partial + L \\ &\iff \quad A^\infty_p + \partial C + F_{p-1}C \leq \ker \partial + F_{p-1}C. \end{aligned}$$

Now $\partial C \leq \ker \partial$ since $\partial$ is a boundary operator, and $A^\infty_p \leq \ker \partial$ by (5).

**Condition (eff)**:

$$\begin{aligned} \operatorname{Im} \iota_p \cap L &= (\partial C + F_{p-1}C) \cap (A^\infty_p + F_{p-1}C) \\ &\overset{(5)}{=} (\partial C \cap F_pC) + F_{p-1}C \\ &\overset{(6)}{=} \partial A^\infty_{p+\infty} + F_{p-1}C \\ &= L_{\iota_p}. \end{aligned}$$

The lifting lemma 4.5 is now applicable, yielding the generalized embeddings $\epsilon_p := \iota_p/\iota$. □

Corollary 4.7 is the special case $m = 2$. In light of Remark 4.6 the theorem thus states that the induced filtration on the total (co)homology is effectively computable, as long as the generalized embeddings $\iota$ and $\iota_p$ are effectively computable for all $p$. Hence, it can be viewed as a (more) constructive version of the **classical convergence theorem** of spectral sequences of filtered complexes, a version that makes use of generalized embeddings:

**Theorem 5.2** (Classical convergence theorem [**Wei94**, Thm. 5.5.1])**.** *Let $C$ be chain complex with a finite filtration $(F_pC)$. Then the associated spectral sequence converges to $H_*(C)$:*

$$E^0_{pq} := F_pC_{p+q}/F_{p-1}C_{p+q} \Longrightarrow H_{p+q}(C).$$

Everything in this section can be reformulated for *co*chain complexes and cohomological spectral sequences.

## 6. Spectral sequences of bicomplexes

Bicomplexes are one of the main sources for filtered complexes in algebra. They are less often encountered in topology. A **homological bicomplex** is a lattice $B = (B_{pq})$ $(p, q \in \mathbb{Z})$ of objects connected with **vertical** morphisms $\partial^{\mathrm{v}}$ pointing *down* and **horizontal** morphisms $\partial^{\mathrm{h}}$ pointing *left*, such that $\partial^{\mathrm{v}}\partial^{\mathrm{h}} + \partial^{\mathrm{h}}\partial^{\mathrm{v}} = 0$.

$$\begin{tikzcd}
B_{02} \arrow[d, "\partial_{\mathrm{v}}"] \arrow[dr, dotted, no head] & B_{12} \arrow[l, "\partial_{\mathrm{h}}"'] \arrow[d, "\partial_{\mathrm{v}}"] \arrow[dr, dotted, no head] & B_{22} \arrow[l, "\partial_{\mathrm{h}}"'] \arrow[d, "\partial_{\mathrm{v}}"] \\
B_{01} \arrow[d, "\partial_{\mathrm{v}}"] \arrow[dr, dotted, no head] & B_{11} \arrow[l, "\partial_{\mathrm{h}}"'] \arrow[d, "\partial_{\mathrm{v}}"] \arrow[dr, dotted, no head] & B_{21} \arrow[l, "\partial_{\mathrm{h}}"'] \arrow[d, "\partial_{\mathrm{v}}"] \\
B_{00} & B_{10} \arrow[l, "\partial_{\mathrm{h}}"'] & B_{20} \arrow[l, "\partial_{\mathrm{h}}"']
\end{tikzcd}$$

$q$ $p$

The **sign trick** $\hat{\partial}_{pq} := (-1)^p \partial^{\mathrm{v}}_{pq}$ converts the **anticommutative** squares into commutative ones, and hence turns the bicomplex into a **complex of complexes** connected with chain maps as morphisms, and vice versa.

The direct sum of objects $\mathrm{Tot}(B)_n := \bigoplus_{p+q=n} B_{pq}$ together with the **total boundary operator** $\partial_n := \sum_{p+q=n} \partial^{\mathrm{v}}_{pq} + \partial^{\mathrm{h}}_{pq}$ form a chain complex called the **the total complex** associated to the bicomplex $B$. $\partial\partial = 0$ is a direct consequence of the anticommutativity.

The vertical morphisms $d_{\mathrm{v}}$ of a **cohomological bicomplex** $(B^{pq})$ point *up* and the horizontal $d_{\mathrm{h}}$ point *right*. We assume all bicomplexes bounded, i.e. only finitely many objects $B_{pq}$ are different from zero.

There exists a natural so-called **column filtration** of the total complex $\mathrm{Tot}(B)$ such that the 0-th page $E^0 = (E^0_{pq}) = (B_{pq})$ of the spectral sequence associated to this filtration consists of the vertical arrows of $B$ and the 1-st page $E^1$ contains morphisms induced by the vertical ones. Its associated spectral sequence is called the **first spectral sequence** of the bicomplex $B$ and is often denoted by ${}^{\mathrm{I}}E$. For a formal definition see [**Wei94**, Def. 5.6.1]. The **second spectral sequence** is the (first) spectral sequence of the **transposed bicomplex** ${}^{\mathrm{tr}}B = ({}^{\mathrm{tr}}B_{pq}) := (B_{qp})$. It is denoted by ${}^{\mathrm{II}}E$. Note that $\mathrm{Tot}(B) = \mathrm{Tot}({}^{\mathrm{tr}}B)$, only the two corresponding filtrations and their induced filtrations on the total cohomology $H_*(\mathrm{Tot}(B))$ differ in general. So the short notation

$$ {}^{\mathrm{I}}E^a_{pq} \Longrightarrow H_{p+q}(\mathrm{Tot}(B)) \Longleftarrow {}^{\mathrm{II}}E^a_{pq} $$

refers in general to two different filtrations of $H_{p+q}(\mathrm{Tot}(B))$.

Here is an algorithm using generalized maps to compute the arrows

$$ \partial^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1} $$

of the $r$-th term of the homological (first) spectral sequence $E^r$. Again, everything can be easily adapted for the cohomological case. Denote by

$$\alpha_S : E^r_{pq} \to B_{pq} \quad \text{resp.} \quad \alpha_T : E^r_{p-r,q+r-1} \to B_{p-r,q+r-1}$$

the generalized embedding of the *s*ource resp. *t*arget of $\partial^r_{pq}$ into the object $B_{pq} = E^0_{pq} \leq \mathrm{Tot}(B)_{p+q}$ resp. $B_{p-r,q+r-1} \leq \mathrm{Tot}(B)_{p+q-1}$. These so-called **absolute embeddings** are the successive compositions of the **relative embeddings** $E^r_{pq} \to E^{r-1}_{pq}$. For the sake of completeness we also mention the **total embeddings**

$$\iota_S : E^r_{pq} \to \mathrm{Tot}(B)_{p+q} \quad \text{resp.} \quad \iota_T : E^r_{p-r,q+r-1} \to \mathrm{Tot}(B)_{p+q-1},$$

the compositions of $\alpha_S$ resp. $\alpha_T$ with the *generalized* embeddings[6] $B_{pq} \to \mathrm{Tot}(B)_{p+q}$ resp. $B_{p-r,q+r-1} \to \mathrm{Tot}(B)_{p+q-1}$.

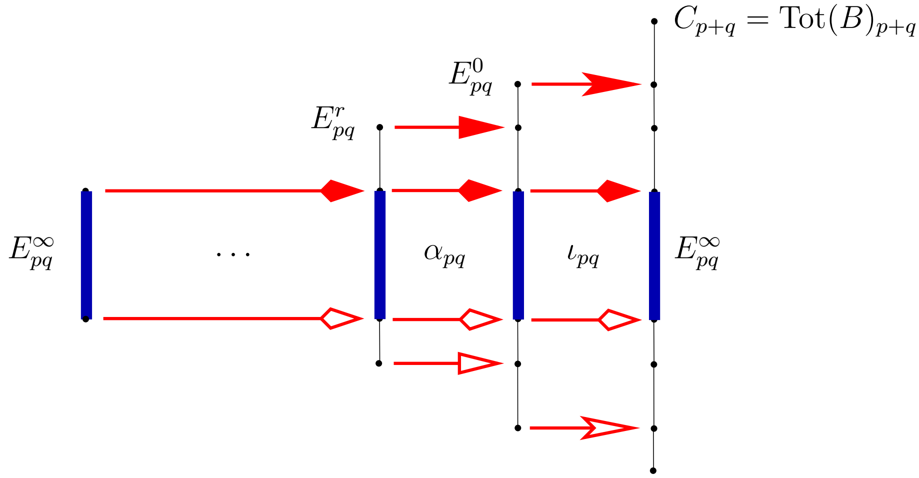



FIGURE 19. The relative, absolute, and total embeddings

For $r > 1$ let

$$h^r_{pq} : B_{pq} \to \bigoplus_{i=1}^{r-1} B_{p-i,q+i-1} \quad \text{and} \quad v^r_{p-r+1,q+r-1} : B_{p-r+1,q+r-1} \to \bigoplus_{i=1}^{r-1} B_{p-i,q+i-1}$$

be the restrictions of the total boundary operator $\partial_{p+q}$ to the specified sources and targets. Similarly, for $r > 2$ let

$$l^r_{pq} : \bigoplus_{i=1}^{r-2} B_{p-i,q+i} \to \bigoplus_{i=1}^{r-1} B_{p-i,q+i-1},$$

[6] It identifies $B_{pq}$ with the *subfactor* of $\mathrm{Tot}(B)_{p+q}$ dictated by the filtration.

again the restriction of the total boundary operator $\partial_{p+q}$ to the specified source and target.

$$\begin{tikzcd}
E^r_{p-r,q+r-1} \arrow[d, hook, "\alpha_T"'] & & & & \\
B_{p-r,q+r-1} & B_{p-r+1,q+r-1} \arrow[l, "\partial^{\mathrm{h}}"'] \arrow[d, "\partial^{\mathrm{v}}"] & & & \\
& B_{p-r+1,q+r-2} \arrow[dr, dotted, no head] & B_{p-r+2,q+r-2} \arrow[l] \arrow[d, dotted] \arrow[dr, dashed, no head] & & \\
& & \ddots \arrow[dr, dotted, no head] & B_{p-1,q+1} \arrow[l, dotted] \arrow[d] & \\
& & & B_{p-1,q} & B_{pq} \arrow[l, "\partial^{\mathrm{h}}"'] \\
& & & & E_{pq} \arrow[u, hook, "\alpha_S"]
\end{tikzcd}$$

We distinguish four cases $r = 0, 1, 2$, and $r > 2$.

$r = 0$: $\partial^0_{pq} := \partial^{\mathrm{v}}_{pq}$. Note that $E^0_{pq} := B_{pq}$.

$r = 1$: $\partial^1_{pq} := \alpha_T^{-1} \circ (\partial^{\mathrm{h}}_{pq} \circ \alpha_S)$.

$r = 2$: $\partial^2_{pq} := \alpha_T^{-1} \circ (\partial^{\mathrm{h}}_{p-1,q+1} \circ (-\beta^{-1} \circ (h^2_{pq} \circ \alpha_S)))$, where $\beta := v^2_{p-1,q+1}$. Note that $h^2_{pq} = \partial^{\mathrm{h}}_{pq}$ and $v^2_{p-1,q+1} = \partial^{\mathrm{v}}_{p-1,q+1}$.

$r > 2$: $\partial^r_{pq} := \alpha_T^{-1} \circ (\partial^{\mathrm{h}}_{p-r+1,q+r-1} \circ (-\beta^{-1} \circ (h^r_{pq} \circ \alpha_S)))$, with $\beta := (v^r_{p-r+1,q+r-1}, l^r_{pq})$, the coarsening of $v^r_{p-r+1,q+r-1}$ with aid $l_{pq}$. We say: $v^r_{p-r+1,q+r-1}$ aided by $l^r_{pq}$ lifts $h^r_{pq} \circ \alpha_S$.

We announced an algorithm and provided closed formulas. This is the true value of generalized maps mentioned in the Introduction. As an easy exercise, the reader might try to rephrase the diagram chasing of the snake lemma as a closed formula in terms of generalized maps. The concept of a generalized map evolved during the implementation of the `homalg` package in GAP [**Bar09**].

It follows from Remark 4.6 that the spectral sequence of a finite type bounded bicomplex (in fact, of a finite type complex with finite filtration) over a computable ring is effectively computable (cf. Def. A.1). The `homalg` package [**Bar09**] contains routines to compute spectral sequences of bicomplexes.

We end this section with a simple example from linear algebra. Let $k$ be a field and $\lambda \in k$ a field element. The JORDAN-form matrix

$$J(\lambda) = \begin{pmatrix} \lambda & 1 & \cdot \\ \cdot & \lambda & 1 \\ \cdot & \cdot & \lambda \end{pmatrix} \in k^{3\times 3}$$

turns $V := k^{1\times 3}$ into a left $k[x]$-module (of finite length), where $x$ acts via $J(\lambda)$, i.e. $xv := J(\lambda)v$ for all $v \in V$. The $k[x]$-module $V$ is filtered and the filtrations stems from a bicomplex:

**Example 6.1** (**Spectrum** of an endomorphism)**.** Let $k$ be a field and $\lambda \in k$. Consider the second quadrant bicomplex $B_\lambda$

$$\begin{tikzcd}
B_{-2,3} \arrow[d, "(x-\lambda)"'] & & \\
B_{-2,2} & B_{-1,2} \arrow[l, "(-1)"'] \arrow[d, "-(x-\lambda)"'] & \\
& B_{-1,1} & B_{0,1} \arrow[l, "(-1)"'] \arrow[d, "(x-\lambda)"'] \\
& & B_{0,0}
\end{tikzcd}$$

with $B_{0,0} = B_{0,1} = B_{-1,1} = B_{-1,2} = B_{-2,2} = B_{-2,3} = k[x]$, all other spots being zero. The total complex contains exactly two nontrivial $k[x]$-modules at degrees 0 and 1 and a single nontrivial morphism

$$\partial_1(\lambda) : \operatorname{Tot}(B)_1 = k[x]^{1\times 3} \longrightarrow k[x]^{1\times 3} = \operatorname{Tot}(B)_0$$

with matrix

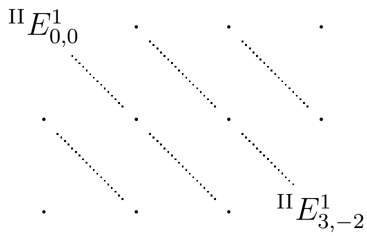
$$x\mathrm{Id} - J(\lambda) = \begin{pmatrix} x-\lambda & -1 & \cdot \\ \cdot & x-\lambda & -1 \\ \cdot & \cdot & x-\lambda \end{pmatrix}.$$

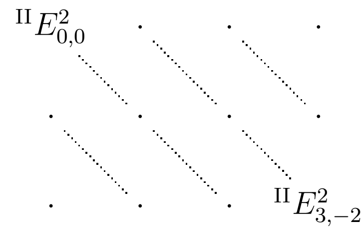
The first spectral sequences ${}^{\mathrm{I}}E$ lives in the second quadrant and stabilizes already at ${}^{\mathrm{I}}E^1 =: {}^{\mathrm{I}}E^\infty$

$$\begin{tikzcd}
\cdot \arrow[dr, dotted, no head] & \cdot \arrow[dr, dotted, no head] & \cdot \\
{}^{\mathrm{I}}E^1_{-2,-2} \arrow[dr, dotted, no head] & \cdot \arrow[dr, dotted, no head] & \cdot \\
\cdot \arrow[dr, dotted, no head] & {}^{\mathrm{I}}E^1_{-1,-1} \arrow[dr, dotted, no head] & \cdot \\
\cdot & \cdot & {}^{\mathrm{I}}E^1_{0,0}
\end{tikzcd}$$

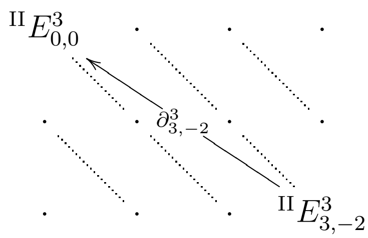
with ${}^{\mathrm{I}}E^\infty_{0,0} = {}^{\mathrm{I}}E^\infty_{-1,-1} = {}^{\mathrm{I}}E^\infty_{-2,-2} = k[x]/\langle x-\lambda\rangle$.

The second spectral sequences ${}^{\mathrm{II}}E$ lives in the fourth quadrant, has only zero arrows at levels 1 and 2

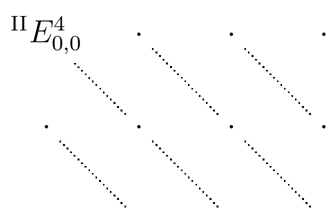



with ${}^{\mathrm{II}}E^1_{0,0} = {}^{\mathrm{II}}E^1_{3,-2} = k[x]$, and hence ${}^{\mathrm{II}}E^2_{0,0} = {}^{\mathrm{II}}E^2_{3,-2} = k[x] = {}^{\mathrm{II}}E^3_{0,0} = {}^{\mathrm{II}}E^3_{3,-2}$. At level 3 there exists a single nonzero arrow $\partial^3_{3,-2}$ with matrix $\big((x-\lambda)^3\big)$:




${}^{\mathrm{II}}E$ finally collapses to its $p$-axes at ${}^{\mathrm{II}}E^4 =: {}^{\mathrm{II}}E^\infty$




with ${}^{\mathrm{II}}E^\infty_{0,0} = k[x]/\langle (x-\lambda)^3\rangle$, providing a spectral sequence proof for the elementary fact that

$$\operatorname{coker} \partial_1(\lambda) \cong k[x]/\langle (x-\lambda)^3\rangle.$$

Conversely, this isomorphism implies that the matrix of the morphism $\partial^3_{3,-2}$ is equal to $\big((x-\lambda)^3\big)$, up to a unit $a \in k^\times$.

## 7. The Cartan-Eilenberg resolution of a complex

The Cartan-Eilenberg **resolution** generalizes the **horse shoe lemma** in the following sense: The horse shoe lemma produces a **simultaneous** projective resolution[7]

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \cdots & 0 \arrow[d] & \\
0 & M' \arrow[l] \arrow[d] & P'_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & P'_d \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & M \arrow[l] \arrow[d] & P_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & P_d \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & M'' \arrow[l] \arrow[d] & P''_0 \arrow[l] \arrow[d] & \cdots \arrow[l] & P''_d \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & 0 & 0 & \cdots & 0 &
\end{tikzcd}$$

of a short exact sequence $0 \leftarrow M'' \leftarrow M \leftarrow M' \leftarrow 0$, where simultaneous means that each row is a projective resolution and all columns are exact. Now let us look at this threefold resolution in the following way: The short exact sequence defines a 2-step filtration of the object $M$ with graded parts $M'$ and $M''$ and the horse shoe lemma states that any resolutions of the graded parts can be put together to a resolution of the total object $M$. In fact, as $P''_i$ is projective, it follows that the total object $P_i$ must even be the direct sum of the graded parts $P'_i$ and $P''_i$. The non-triviality of the filtration on $M$ is reflected in the fact that the morphisms of the total resolution $P_*$ are in general not merely the direct sum of the morphisms in the resolutions $P'_*$ and $P''_*$ of the graded parts $M'$ and $M''$. This statement can now be generalized to $m$-step filtrations simply by applying the (2-step) horse shoe lemma inductively.

Now consider a complex $(C, \partial)$, which is not necessarily exact. On each object $C_n$ the complex structure induces a 3-step filtration $0 \leq B_n \leq Z_n \leq C_n$, with boundaries $B_n := \operatorname{im} \partial_{n+1}$ and cycles $Z_n := \ker \partial_n$. The above discussion now applies to the three graded parts $B_n$, $H_n := Z_n/B_n$ and $C_n/Z_n$ and any three resolution thereof can be put together to a resolution of the total object $C_n$. If one takes into account the fact that $\partial_{n+1}$ induces an isomorphism between $C_{n+1}/Z_{n+1}$ and $B_n$ (for all $n$, by the homomorphism theorem), then all total resolutions of all the $C_n$'s can be constructed in a compatible way so that they fit together in one complex of complexes. This complex is called the Cartan-Eilenberg resolution of the complex $C$.

A formal version of the above discussion can be found in [**HS97**, Lemma 9.4] or [**Wei94**, Lemma 5.7.2]. Since the projective horse shoe lemma is constructive, the projective Cartan-Eilenberg resolution is so as well.

[7] We will only refer to projective resolutions as they are more relevant to effective computations.

## 8. GROTHENDIECK's spectral sequences

Let $\mathcal{C} \stackrel{F}{\leftarrow} \mathcal{B} \stackrel{G}{\leftarrow} \mathcal{A}$ be composable functors of abelian categories. The so-called GROTHENDIECK spectral sequence relates, under mild assumptions, the composition of the derivations of $F$ and $G$ with the derivation of their composition $F \circ G$. There are 16 versions of the GROTHENDIECK spectral sequence, depending on whether $F$ resp. $G$ is co- or contravariant, and whether $F$ resp. $G$ is being left or right derived. Four of them do not use injective resolutions and are therefore rather directly accessible to a computer. In this section two versions out of the four are reviewed: The filtrations of $L \otimes_D M$ and $\mathrm{Hom}_D(M, N)$ mentioned in the Introduction are recovered in the next section as the spectral filtrations induced by these two GROTHENDIECK spectral sequences, after appropriately choosing the functors $F$ and $G$.

**Theorem 8.1** (GROTHENDIECK spectral sequence, [**Rot79**, Thm. 11.41])**.** *Let $F$ and $G$ be contravariant functors and let every object in $\mathcal{A}$ and $\mathcal{B}$ has a* finite *projective resolution. Under the assumptions that*

(1) *$G$ maps projective objects to $F$-acyclic objects and that*
(2) *$F$ is left exact,*

*then there exists a second quadrant homological spectral sequence with*

$$E^2_{pq} = \mathrm{R}^{-p} F \circ \mathrm{R}^q G \Longrightarrow \mathrm{L}_{p+q}(F \circ G).$$

PROOF. Let $M$ be an object in $\mathcal{A}$ and $P_\bullet = (P_p)$ a finite projective resolution of $M$. Denote by $CE = (CE^{p,q})$ the projective CARTAN-EILENBERG resolution of the cocomplex $(Q^p) := (G(P_p))$. It exists since $\mathcal{B}$ has enough projectives. Note that $q \leq 0$ since $CE$ is a cohomological bicomplex. Define the homological bicomplex $B = (B_{p,q}) := (F(CE^{p,q}))$. We call $B$ the **Grothendieck bicomplex** associated to $M$, $F$, and $G$. It lives in the fourth quadrant and is bounded in both directions.

**The first spectral sequence ${}^{\mathrm{I}}E$:**
For fixed $p$ the vertical cocomplex $CE^{p,\bullet}$ is, by construction, a projective resolution of $G(P_p)$. Hence ${}^{\mathrm{I}}E^1_{pq} = \mathrm{R}^{-q} F(G(P_p))$. But since $G(P_p)$ is $F$-acyclic by assumption (1), the first sheet collapses to the 0-th row. The left exactness of $F$ implies that $R^0 F = F$ and hence ${}^{\mathrm{I}}E^1_{p0} = (F \circ G)(P_p)$. I.e. the 0-th row of ${}^{\mathrm{I}}E^1$ is nothing but the covariant functor $F \circ G$ applied to the projective resolution $(P_p)$ of $M$. The first spectral sequences of $B$ thus stabilizes at level 2 with the single row ${}^{\mathrm{I}}E^2_{n,0} = \mathrm{L}_n(F \circ G)(M)$.

**The second spectral sequence ${}^{\mathrm{II}}E$:**
The second spectral sequence of the bicomplex $B$ is by definition the spectral sequence of its transposed $({}^{\mathrm{tr}}B_{pq}) := (B_{qp})$, a second quadrant bicomplex. Obviously ${}^{\mathrm{tr}}B = F({}^{\mathrm{tr}}CE)$. By definition, the $q$-th row ${}^{\mathrm{II}}E^1_{\bullet,q} := H^{\mathrm{vert}}_{\bullet,q}({}^{\mathrm{tr}}B) = H^{\mathrm{vert}}_{\bullet,q}(F({}^{\mathrm{tr}}CE)) = F(H^{\bullet,q}_{\mathrm{vert}}({}^{\mathrm{tr}}CE))$, where the last equality follows from the properties of the CARTAN-EILENBERG resolution and the additivity of $F$. Now recall that the vertical cohomologies $H^{\bullet,q}_{\mathrm{vert}}({}^{\mathrm{tr}}CE)$ are for fixed $q$, again by construction, projective resolutions of the cohomology $H^q(G(P_\bullet)) =: \mathrm{R}^q G(M)$. Hence ${}^{\mathrm{II}}E^2_{pq} = \mathrm{R}^{-p} F(\mathrm{R}^q G(M))$. $\square$

The proof shows that assumptions (1) and (2) only involve the first spectral sequence. Assumption (1) guaranteed the collapse of the first spectral sequence at the first level, while (2) ensures that the natural transformation $F \to \mathrm{R}^0 F$ is an equivalence. In other words, dropping (2) means replacing $\mathrm{L}_{p+q}(F \circ G)$ by $\mathrm{L}_{p+q}(\mathrm{R}^0 F \circ G)$.

**Theorem 8.2** (Grothendieck spectral sequence)**.** *Let $F$ be a covariant and $G$ a contravariant functor and let every object in $\mathcal{A}$ and $\mathcal{B}$ has a* finite *projective resolution. Under the assumptions that*

(1) *$G$ maps projective objects to $F$-acyclic objects and that*
(2) *$F$ is right exact,*

*then there exists a second quadrant cohomological spectral sequence with*

$$E^2_{pq} = \mathrm{L}_{-p} F \circ \mathrm{R}^q G \Longrightarrow \mathrm{R}^{p+q}(F \circ G).$$

Proof. Again the first spectral sequence is a fourth quadrant spectral sequence while the second lives in the second quadrant. Assumption (2) ensures that the natural transformation $\mathrm{L}^0 F \to F$ is an equivalence. The above proof and the subsequent remark can be copied with the obvious modifications. □

**Remark 8.3** (One sided boundedness)**.** The existence of finite projective resolutions in $\mathcal{A}$ and $\mathcal{B}$ led the spectral sequences to be bounded in both directions. In order to avoid convergence subtleties it would suffice to assume boundedness in just one direction by requiring that either $\mathcal{A}$ or $\mathcal{B}$ allows finite projective resolutions while the other has enough projectives. The assumption of the existence of *finite* projective resp. injective resolutions can be dropped when dealing with the versions of the Grothendieck spectral sequences that live in the first resp. third quadrant.

## 9. Applications

This section recalls how the natural filtrations mentioned in examples (a), (a'), and (d) of the Introduction can be recovered as **spectral filtrations**.

Theorems 8.1 and 8.2 admit an obvious generalization. The composed functor $F \circ G$ can be replaced by a functor $H$ that coincides with $F \circ G$ on projectives (for other versions of the Grothendieck spectral sequence the "projectives" has to be replaced by "injectives"). As usual, $D$ is an associative ring with 1. $\mathrm{Ext}^n_D$ and $\mathrm{Tor}^D_n$ are abbreviated as $\mathrm{Ext}^n$ and $\mathrm{Tor}_n$.

**Assumption:** In this section the left *or* right global dimension[8] of $D$ is assumed finite. The involved spectral sequences will then be bounded in (at least) one direction (see Remark 8.3).

[8] Recall, the **left global (homological) dimension** is the supremum over all projective dimensions of *left* $D$-modules (see Subsection 9.1.5). If $D$ is left Noetherian, then the left global dimension of $D$ coincides with the **weak global (homological) dimension**, which is the largest integer $\mu$ such that $\mathrm{Tor}^D_\mu(M, N) \neq 0$ for some right module $M$ and left module $N$, otherwise infinity (cf. [**MR01**, 7.1.9]). This last definition is obviously left-right symmetric. The same is valid if "left" is replaced by "right".

### 9.1. The double-Ext spectral sequence and the filtration of Tor.

**Corollary 9.1** (The double-Ext spectral sequence). *Let $M$ be a left $D$-module and $L$ a right $D$-module. Then there exists a second quadrant homological spectral sequence with*

$$E^2_{pq} = \operatorname{Ext}^{-p}(\operatorname{Ext}^q(M,D),L) \Longrightarrow \operatorname{Tor}_{p+q}(L,M).$$

*In particular, there exists an ascending filtration of $\operatorname{Tor}_{p+q}(L,M)$ with $\operatorname{gr}_p \operatorname{Tor}_{p+q}(L,M)$ naturally isomorphic to a subfactor of $\operatorname{Ext}^{-p}(\operatorname{Ext}^q(M,D),L)$, $p \leq 0$.*

The special case $p+q=0$ recovers the filtration of $L \otimes M$ mentioned in Example (a) of the Introduction via the natural isomorphism $L \otimes M \cong \operatorname{Tor}_0(L,M)$.

9.1.1. *Using the* GROTHENDIECK *bicomplex.* Corollary 9.1 is a consequence of Theorem 8.1 for $F := \operatorname{Hom}_D(-,L)$ and $G := \operatorname{Hom}_D(-,D)$, since $F \circ G$ coincides with $L \otimes_D -$ on projectives.

To be able to effectively compute double-Ext (groups in) the GROTHENDIECK bicomplex the ring $D$ must be computable in the sense that *two* sided inhomogeneous linear systems $A_1 X_1 + X_2 A_2 = B$ must be effectively solvable, where $A_1$, $A_2$, and $B$ are matrices over $D$ (see [**BR08**, Subsection 6.2.4]). This is immediate for computable commutative rings (cf. Def. A.1). In B.2 an example over a commutative ring is treated.

9.1.2. *Using the bicomplex $I_L \otimes P^M$.* The **bifunctoriality** of $\otimes$ leads to the following homological bicomplex

$$B := I_L \otimes P^M \cong \operatorname{Hom}(\operatorname{Hom}(P^M,D),I_L),$$

where $P^M$ is an injective resolution of $M$ and $I_L$ is an injective resolution of $L$. Starting from $r=2$ the first and second spectral sequence of $B$ coincide with those of the GROTHENDIECK bicomplex associated to $M$, $F := \operatorname{Hom}_D(-,L)$, and $G := \operatorname{Hom}_D(-,D)$. In contrast to the GROTHENDIECK bicomplex the bicomplex $B$ is over most of the interesting rings in general highly nonconstructive as an injective resolution enters its definition. In [**HL97**, Lemma 1.1.8] a sheaf variant of this bicomplex was used to "compute" the purity filtration (see below).

9.1.3. *The bidualizing complex.* Taking $L = D$ as a right $D$-module in Corollary 9.1 recovers the **bidualizing spectral sequence** of J.-E. ROOS [**Roo62**].

$$E^2_{pq} = \operatorname{Ext}^{-p}(\operatorname{Ext}^q(M,D),D) \Longrightarrow \begin{cases} M & \text{for } p+q=0, \\ 0 & \text{otherwise.} \end{cases}$$

The GROTHENDIECK bicomplex is then known as the **bidualizing complex**. The case $p+q=0$ defines the **purity filtration**[9] $(\mathrm{t}_{-c} M)$ of $M$, which was motivated in Example (a') of the Introduction. For more details cf. [**Bjö79**, Chap. 2, §5,7].

The module $M_c = E^\infty_{-c,c}$ is for $c=0$ and $c=1$ a submodule of $\operatorname{Ext}^c(\operatorname{Ext}^c(M,D),D) = E^2_{-c,c}$ and for $c \geq 2$ in general only a subfactor. All this is obvious from the shape of the bidualizing spectral sequence.

[9] Unlike [**Bjö79**, Chap. 2, Subsection 4.15], we only make the weaker assumption stated at the beginning of the section.

Since $M_c = \operatorname{t}_{-c} M / \operatorname{t}_{-(c+1)} M$ it follows that the **higher evaluations maps** $\varepsilon_c$

$$0 \to \operatorname{t}_{-(c+1)} M \to \operatorname{t}_{-c} M \xrightarrow{\varepsilon_c} \operatorname{Ext}^c_D(\operatorname{Ext}^c_D(M,D),D)$$

mentioned in the Introduction are only a different way of writing the generalized embeddings

$$\bar{\varepsilon}_c : M_c \to \operatorname{Ext}^c(\operatorname{Ext}^c(M,D),D).$$

So without further assumptions $\varepsilon_c$ (resp. $\bar{\varepsilon}_c$) is known to be an ordinary morphism (resp. embedding) only for $c=0$ and $c=1$. Now assuming that $E^2_{pq} := \operatorname{Ext}^{-p}(\operatorname{Ext}^q(M,D),D)$ vanishes[10] for $p+q=1$, then all arrows ending at total degree $p+q=0$ vanish (as they all start at total degree $p+q=1$). It follows that for all $c$ the module $M_c$ is not merely a subfactor of $\operatorname{Ext}^c(\operatorname{Ext}^c(M,D),D)$ but a submodule, or, equivalently, $\varepsilon_c$ (resp. $\bar{\varepsilon}_c$) is an ordinary morphism (resp. embedding).

In any case the module $\operatorname{Ext}^c(\operatorname{Ext}^c(M,D),D)$ is called the **reflexive hull** of the **pure** subfactor $M_c$.

**Definition 9.2** (Pure, reflexively pure)**.** A module $M$ is called **pure** if it consists of exactly one nontrivial pure subfactor $M_c$ or is zero. A nontrivial module $M$ is called **reflexively pure** if it is pure and if the generalized embedding $M = M_c \to \operatorname{Ext}^c(\operatorname{Ext}^c(M,D),D)$ is an isomorphism. Define the zero module to be reflexively pure.

If $M$ is a finitely generated $D$-module, then all ingredients of the bidualizing complex are again finitely generated (projective) $D$-modules, even if the ring $D$ is *non*commutative. It follows that the purity filtration over a computable ring $D$ is effectively computable. A commutative and a noncommutative example are given in B.3 and B.4 respectively. The latter demonstrates how the purity filtration (as a filtration that always exists) can be used to transform a linear system of PDEs into a triangular form where now a cascade integration strategy can be used to obtain exact solutions. The idea of viewing a linear system of PDEs as a module over an appropriate ring of differential operators was emphasized by B. Malgrange in the late 1960's and according to him goes back to Emmy Noether.

9.1.4. *Criterions for reflexive purity.* This subsection lists some simple criterions for reflexive purity of modules.

First note that the existence of the bidualizing spectral sequence immediately implies that the set $c(M) := \{c \geq 0 \mid \operatorname{Ext}^c_D(M,D) \neq 0\}$ is empty only if $M=0$. Recall that if $c(M)$ is nonempty, then its minimum is called the **grade** or **codimension** of $M$ and denoted by $j(M)$ or $\operatorname{codim} M$. The codimension of the zero module is set to be $\infty$. Further define $\bar{q}(M) := \sup c(M)$ in case $c(M) \neq \emptyset$, and $\infty$ otherwise.

All of the following arguments make use of the shape of the bidualizing spectral sequence in the respective situation.

• If $c(M)$ contains a single element, i.e. if $\operatorname{codim} M = \bar{q}(M) =: \bar{q} < \infty$, then $M = M_{\bar{q}}$ is reflexively pure of codimension $\bar{q}$, giving a simple spectral sequence proof of [**Qua01**, Thm. 7].

[10]This condition is satisfied for an Auslander **regular** ring $D$: $\operatorname{Ext}^{-p}(\operatorname{Ext}^q(M,D),D) = 0$ for all $p+q>0$ and all $D$-modules $M$. See [**Bjö79**, Chap. 2: Cor. 5.18, Cor. 7.5].

For the remaining criterions assume that $\operatorname{Ext}^{-p}(\operatorname{Ext}^q(M,D),D)=0$ for $p+q=1$:

• If $\bar{q} := \bar{q}(M)$ is finite, then $E^2_{-\bar{q},\bar{q}} = E^\infty_{-\bar{q},\bar{q}}$, i.e. $M_{\bar{q}}$ is reflexively pure (possibly zero). This generalizes the above criterion (under the assumption just made).

• Now if $M$ is a left (resp. right) $D$-module, then assume further that the right (resp. left) global dimension $d$ of the ring $D$ is finite. It follows that $E^2_{-c,c} = E^\infty_{-c,c}$ for $c=d$ and $c=d-1$. This means that under the above assumptions the subfactors $M_d$ and $M_{d-1}$ are always reflexively pure[11].

9.1.5. *Codegree of purity.* As a GROTHENDIECK spectral sequence the bidualizing spectral sequence becomes intrinsic at level 2. Each $E^2_{-c,c}$ starts to "shrink" until it stabilizes at $E^\infty_{-c,c} = M_c$. Motivated by this define the **codegree of purity** $\operatorname{cp} M$ of a module $M$ as follows: Set $\operatorname{cp} M$ to $\infty$ if $M$ is not pure. Otherwise $\operatorname{cp} M$ is a tuple of nonnegative integers, the length of which is one plus the number of times $E^a_{-c,c}$ shrinks (nontrivially[12]) for $a \geq 2$ until it stabilizes at $M_c$. The entries of this tuple are the numbers of pages between the drops, i.e. the width of the steps in the staircase of objects $(E^a_{-c,c})_{c\geq 2}$. It follows that the sum over the entries of $\operatorname{cp} M$ is the number of pages it takes for $E^2_{-c,c}$ until it reaches $M_c$. In particular, a module is *reflexively pure* if and only if $\operatorname{cp} M = (0)$.

The codegree of purity appears in Examples B.3 and B.4. In Example B.7 the codegree of purity is compared with two other classical homological invariants:
Recall, the **projective dimension** of a module $M$ is defined to be the length $d$ of the shortest projective resolution $0 \leftarrow M \leftarrow P_0 \leftarrow \cdots \leftarrow P_d \leftarrow 0$. AUSLANDER**'s degree of torsion-freeness** of a module $M$ is defined following [**AB69**, Def. on p. 2 & Def. 2.15(b)] to be the smallest *nonnegative* integer $i$, such that $\operatorname{Ext}^{i+1}(\mathrm{A}(M),D) \neq 0$, otherwise $\infty$, where $\mathrm{A}(M)$ is the so-called AUSLANDER **dual** of $M$ (see also [**Qua01**, Def. 5], [**CQR05**, Def. 19]). To construct $\mathrm{A}(M)$ take a projective presentation $0 \leftarrow M \leftarrow P_0 \xleftarrow{d_1} P_1$ of $M$ and set

$$\mathrm{A}(M) := \operatorname{coker}(P_0^* \xrightarrow{d_1^*} P_1^*),$$

where $d_1^* := \operatorname{Hom}(d_1,D)$ (cf. [**AB69**, p. 1 & Def. 2.5]). Like the syzygies modules, it is proved in [**AB69**, Prop. 2.6(b)] that $\mathrm{A}(M)$ is uniquely determined by $M$ up to **projective equivalence** (see also [**Qua99**] and [**PQ00**, Thm. 2]). In particular, the degree of torsion-freeness is well-defined. The fundamental exact sequence [**AB69**, (0.1) & Prop. 2.6(a)]

$$0 \to \operatorname{Ext}^1_D(\mathrm{A}(M),-) \to M \otimes_D - \to \operatorname{Hom}_D(M^*,-) \to \operatorname{Ext}^2_D(\mathrm{A}(M),-) \to 0,$$

applied to $D$, characterizes torsion-freeness and reflexivity of the module $M$ (see also [**HS97**, Exercise IV.7.3], [**CQR05**, Thm. 6]). For a characterization of projectivity using the degree of torsion-freeness see [**CQR05**, Thm. 7].

The codegree of purity can be defined for quasi-coherent sheaves of modules replacing $D$ by the structure sheaf $\mathcal{O}_X$ or by the dualizing sheaf[13] if it exists. It is important to note that the codegree of purity of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a projective scheme

[11] In case $D = A_n$, the $n$-th WEYL algebra over a field, this says that **holonomic** and **subholonomic** modules are reflexively pure. See [**Bjö79**, Chap. 2, §7].

[12] i.e. passes to a *true* subfactor.

[13] It may even be defined for objects in an abelian category with a dualizing object.

$X = \operatorname{Proj}(S)$ may differ from the codegree of purity of a graded $S$-module $M$ used to represent $\mathcal{F} = \widetilde{M} = \operatorname{Proj} M$. This is mainly due to the fact that $\mathcal{F} = \widetilde{M}$ vanishes for ARTINian modules $M$.

There are several obvious ways how one can refine the codegree of purity to get sharper invariants. The codegree of purity is an example of what can be called a **spectral invariant**.

### 9.2. The Tor-Ext spectral sequence and the filtration of Ext.

**Corollary 9.3** (The Tor − Ext spectral sequence)**.** *Let $M$ and $N$ be left $D$-modules. Then there exists a second quadrant cohomological spectral sequence with*

$$E_2^{pq} = \operatorname{Tor}_{-p}(\operatorname{Ext}^q(M, D), N) \Longrightarrow \operatorname{Ext}^{p+q}(M, N).$$

*In particular, there exists a descending filtration of $\operatorname{Ext}^{p+q}(M, N)$ with $\operatorname{gr}^p \operatorname{Ext}^{p+q}(M, N)$ naturally isomorphic to a subfactor of $\operatorname{Tor}_{-p}(\operatorname{Ext}^q(M, D), N)$, $p \leq 0$*

The special case $p + q = 0$ recovers the filtration of $\operatorname{Hom}(M, N)$ mentioned in Example (d) of the Introduction via the natural isomorphism $\operatorname{Hom}(M, N) \cong \operatorname{Ext}^0(M, N)$.

For **holonomic** modules $M$ over the WEYL $k$-algebra $D := A_n$ the special case formula

$$\operatorname{Hom}(M, N) \cong \operatorname{Tor}_n(\operatorname{Ext}^n(M, D), N)$$

(cf. [**Bjö79**, Chap. 2, Thm. 7.15]) was used by H. TSAI and U. WALTHER in the case when also $N$ is holonomic to compute the finite dimensional $k$-vector space of homomorphisms [**TW01**].

The induced filtration on $\operatorname{Ext}^1(M, N)$ can be used to attach a numerical invariant to each extension of $M$ with submodule $N$. This gives another example of a **spectral invariant**.

9.2.1. *Using the* GROTHENDIECK *bicomplex.* Corollary 9.3 is a consequence of Theorem 8.2 for $F := - \otimes_D N$ and $G := \operatorname{Hom}_D(-, D)$ since $F \circ G$ coincides with $\operatorname{Hom}_D(-, N)$ on projectives. See Example B.5.

9.2.2. *Using the bicomplex* $\operatorname{Hom}(P^M, P^N)$*.* The **bifunctoriality** of Hom leads to the following cohomological bicomplex

$$B := \operatorname{Hom}(P^M, P^N) \cong \operatorname{Hom}(P^M, D) \otimes P^N,$$

where $P^L$ denotes a projective resolution of the module $L$. It is an easy exercise (cf. [**Bjö79**, Chap. 2, §4.14]) to show that starting from $r = 2$ the first and second spectral sequence of $B$ coincide with those of the GROTHENDIECK bicomplex associated to $M$, $F := - \otimes_D N$ and $G := \operatorname{Hom}_D(-, D)$. Both bicomplexes are constructive as only projective resolutions enter their definitions. The bicomplex $B$ has the computational advantage of avoiding the rather expensive CARTAN-EILENBERG resolution used to define the GROTHENDIECK bicomplex. See Example B.6. Compare the output of the command `homalgRingStatistics` in Example B.6 with corresponding output in Example B.5.

Since the first spectral sequence of the bicomplex $B := \operatorname{Hom}(P^M, P^N)$ collapses a small part of it is often used to compute $\operatorname{Hom}(M, N)$ over a *commutative* ring $D$, as then all arrows of $B$ are again morphisms of $D$-modules. See [**GP02**, p. 104] and [**BR08**, Subsection 6.2.3].

If the ring $D$ is *not* commutative, then the above bicomplex and the GROTHENDIECK bicomplex in the previous subsection fail to be $D$-bicomplexes (unless when $M$ or $N$ is a $D$-bimodule). The bicomplexes are even in a lot of applications of interest *not* of finite type over their natural domain of definition. In certain situations there nevertheless exist *quasi-isomorphic* subfactor (bi)complexes which can be used to perform effective computations. In [**TW01**], cited above, and in the pioneering work [**OT01**] KASHIWARA's so-called $V$-filtration is used to extract such subfactors.

# Simplicial Cohomology of Orbifolds Revisited

This chapter is joint work with SIMON GÖRTZEN. His `homalg` based package `SCO` [**Gör08**] provides the computational tool underlying this work (see also [**GÖ8**]).

## 10. Introduction

Orbifolds are among the so-called generalized spaces where the two concepts of space and symmetry come together. And like most of these spaces, orbifolds can be regarded as a groupoid, which is simply a small category with invertible arrows. That's all. Groupoids are the most natural models for generalized spaces. They were implicitly used by SOPHUS LIE, but in an extensive manner. Only the name was missing. BRANDT gave them their definite name "groupoids". In the hands of ALEXANDER GROTHENDIECK in algebraic geometry and the Bangor group headed by RONNY BROWN in algebraic topology and higher category theory they became one of the most fundamental objects in modern mathematics. Modern schools like the noncommutative geometry school lead by ALAIN CONNES and the symplectic geometry school lead by ALAN WEINSTEIN further deepened the importance of groupoids in geometry.

The groupoid underlying an orbifold is only uniquely defined up to some equivalence relation. This gives one the incredible freedom to travel between very different looking groupoids without altering the orbifold. The work of [**MP99**] shows that under some mild conditions on the orbifold $\mathcal{M}$, there exists a groupoid modelling $\mathcal{M}$ which can be described in a purely combinatorial way using so-called simplical sets. Using spectral sequence arguments one can now transfer difficult cohomological questions about $\mathcal{M}$ to their cohomological counterpart on the simplical set, and finally this allows computations. More precisely to an orbifold $\mathcal{M}$ we will use the notion of groupoids to show how to construct a simplicial set $S_\bullet(\mathcal{M})$ associated to $\mathcal{M}$ with

$$H^*(\mathcal{M},\mathcal{A}) = H^*(S_\bullet(\mathcal{M}), A).$$

## 11. Orbifolds

**Definition 11.1** (Orbifold [**MP99**, Def. 1.1])**.** Let $M$ be a paracompact HAUSDORFF space. An *orbifold chart* on $M$ is given by a connected open subset $\tilde{U} \subseteq \mathbb{R}^n$ for some integer $n \geq 0$, a finite group $G$ of $C^\infty$-automorphisms of $\tilde{U}$, and a map $\varphi : \tilde{U} \to M$, such that $\varphi$ is $G$-invariant ($\varphi \circ g = \varphi$ for all $g \in G$) and induces a homeomorphism of $\tilde{U}/G$ onto the open subset $U = \varphi(\tilde{U}) \subseteq M$. An *embedding* $\lambda : (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ between two such charts is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ with $\psi \circ \lambda = \varphi$. An *orbifold atlas* on

$M$ is a family $\mathcal{U} = \{(\tilde{U}, G, \varphi)\}$ of such charts, which cover $M$ and are locally compatible in the following sense: given any two charts $(\tilde{U}, G, \varphi)$ for $U = \varphi(\tilde{U}) \subseteq M$ and $(\tilde{V}, H, \psi)$ for $V \subseteq M$, and a point $x \in U \cap V$, there exists an open neighborhood $W \subseteq U \cap V$ of $x$ and a chart $(\tilde{W}, K, \chi)$ for $W$ such that there are embeddings $(\tilde{W}, K, \chi) \hookrightarrow (\tilde{U}, G, \varphi)$ and $(\tilde{W}, K, \chi) \hookrightarrow (\tilde{V}, H, \psi)$. Two such atlases are said to be equivalent if they have a common refinement. An *orbifold* (of dimension $n$) is such a space $M$ with an equivalence class of atlases $\mathcal{U}$. We will generally write $\mathcal{M} = (M, \mathcal{U})$ for the orbifold $\mathcal{M}$ represented by the space $M$ and a chosen atlas $\mathcal{U}$.

**Example 11.2.** The main example of [**MP99**] is the $C_n$-teardrop orbifold, where $C_n$ is the cyclic group of order $n$. Here the underlying space is $S^2$, the 2-sphere with its natural topology. Consider the following orbifold atlas $\mathcal{U}$ consisting of six orbifold charts: Two charts $\tilde{U}$ and $\tilde{L}$, which cover open neighborhoods $U$ and $L$ of the upper and lower hemisphere, respectively. $\tilde{U}$ is the only chart with a non-trivial group action; we take $\tilde{U}$ an open disk on which $C_n$ acts by centered rotations. To satisfy the compatibility condition we further need at least two charts $\tilde{E}_1$ and $\tilde{E}_2$, covering open neighborhoods $E_1$ and $E_2$ of the two halfs of the equator. The intersection of $E_1$ and $E_2$ is the disjoint union of the two connected open sets $W_1$ and $W_2$. The last two charts are charts $\tilde{W}_1$ and $\tilde{W}_2$ covering $W_1$ and $W_2$, respectively. [**MP99**, Subsection 3.1] specify an atlas $\mathcal{U}$, where the equator is instead covered by three charts together with their three intersections. There they first construct a triangulation $\mathcal{T}$ of the teardrop orbifold and define $\mathcal{U}$ such that $\mathcal{T}$ is adapted to $\mathcal{U}$ (cf. Definition 13.3).

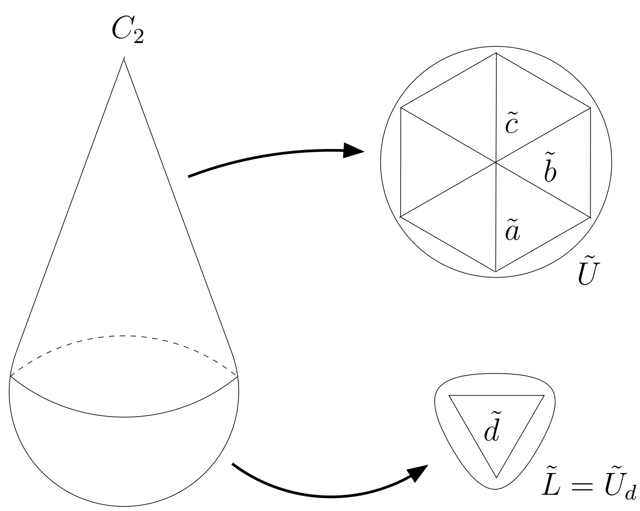



FIGURE 20. The $C_2$-teardrop orbifold.

Now we recall the definition of a sheaf on an orbifold $\mathcal{M}$.

**Definition 11.3.** A sheaf of abelian groups on $\mathcal{M} = (M, \mathcal{U})$ is given by

(1) a sheaf of abelian groups $\mathcal{A}_{\tilde{U}}$ for each $(\tilde{U}, G_U, \varphi_U) \in \mathcal{U}$;

(2) an isomorphism $\mathcal{A}(\lambda) : \mathcal{A}_{\tilde{U}} \to \lambda^*\mathcal{A}_{\tilde{V}}$ for each embedding $\lambda : (\tilde{U}, G_U, \varphi_U) \to (\tilde{V}, G_V, \varphi_V)$ satisfying the "chain rule"

$$\begin{tikzcd} \mathcal{A}_{\tilde{U}} \arrow[r, "\mathcal{A}(\lambda)"] \arrow[d, "\mathcal{A}(\eta\lambda)"'] \arrow[dr, phantom, "\circlearrowleft"] & \lambda^*\mathcal{A}_{\tilde{V}} \arrow[d, "\lambda^*\mathcal{A}(\eta)"] \\ (\eta\lambda)^*\mathcal{A}_{\tilde{W}} \arrow[r, equal, "\text{can.}"] & \lambda^*\eta^*\mathcal{A}_{\tilde{W}} \end{tikzcd}$$

where $\eta : (\tilde{V}, G_V, \varphi_V) \to (\tilde{W}, G_W, \varphi_W)$ is another embedding.

Obviously $\mathcal{A}_{\tilde{U}}$ is a $G_U$-equivariant sheaf on $\tilde{U}$. The sheaves on $\mathcal{M}$, together with their morphisms, form an abelian category $\mathrm{Ab}(\mathcal{M})$ with enough injectives. By definition, a global section is given by sections $s_{\tilde{U}} \in \Gamma(\tilde{U}, \mathcal{A}_{\tilde{U}})$, for each chart $\tilde{U}$, such that $\mathcal{A}(\lambda)(s_{\tilde{U}}) = \lambda^*(s_{\tilde{V}})$ for any embedding $\lambda : \tilde{U} \to \tilde{V}$. The abelian group of all global sections in $\mathcal{A}$ is denoted by $\Gamma(\mathcal{M}, \mathcal{A})$. The $n$-th cohomology of the orbifold $\mathcal{M}$ with values in the sheaf $\mathcal{A}$ is defined as the $n$-th derived functor of the left exact global section functor $\Gamma$:

$$H^n(\mathcal{M}, \mathcal{A}) = (\mathrm{R}^n\,\Gamma)(\mathcal{M}, \mathcal{A}).$$

## 12. From Orbifold to Groupoid

In [**Hae84**] Haefliger associates to each orbifold $\mathcal{M} = (M, \mathcal{U})$ an étale proper topological groupoid $H$ built as follows:

$H_0 := \coprod_{\tilde{U} \in \mathcal{U}} \tilde{U}$ as a topological space. Each point in $H_0$ can be addressed as a pair $(\tilde{x}, \tilde{U})$ with $\tilde{x} \in \tilde{U}$. Each arrow $g : (\tilde{x}, \tilde{U}) \to (\tilde{y}, \tilde{V})$ is an equivalence class of triples $[\lambda, \tilde{z}, \eta] : \tilde{U} \xleftarrow{\lambda} \tilde{W} \xrightarrow{\eta} \tilde{V}$, where $\tilde{z} \in \tilde{W}$ and $\lambda(\tilde{z}) = \tilde{x}$, $\eta(\tilde{z}) = \tilde{y}$. Here $\tilde{W}$ is another chart for $\mathcal{M}$, and $\lambda$, $\eta$ are embeddings. By definition of the embeddings it follows that $\varphi_U(\tilde{x}) = \varphi_V(\tilde{y})$, i.e. that $\tilde{x}$ and $\tilde{y}$ lie over the same point in $M$. The equivalence relation is generated by $[\lambda, \tilde{z}, \eta] = [\lambda\nu, \tilde{z}', \eta\nu]$, for $\lambda, \tilde{z}, \eta$ as above and $\nu : \tilde{W}' \to \tilde{W}$ another embedding, with $\nu(\tilde{z}') = \tilde{z}$.

**Definition 12.1** (Haefliger groupoid)**.** The groupoid $H$ constructed above is called the Haefliger *groupoid* of the orbifold $\mathcal{M} = (M, \mathcal{U})$.

There is a natural topology on the set of arrows $H_1$ of the Haefliger groupoid, that turns it into an étale proper groupoid ([**Hae84, Pro95**]).

If all isotropy groups vanish then $\mathcal{M}$ is a manifold and the Haefliger groupoid specializes to the construction in [**Con94**, II.2.$\alpha$]

Figure 21 shows two bands of arrows $B$ and $B'$ which we view as two disjoint subsets of $H_1$. They have the same set of sources $s(B) = s(B')$, which, as a subspace of $H_0$, is closed and has the topology of a (compact) 1-simplex. In the natural topology of $H_1$ each of the two bands is a closed subspace of $H_1$ having the topology of a 1-simplex. Moreover, $B$ and $B'$ are seperated in $H_1$. In Figure 22 the two bands of arrows travel from one chart to another. Each of the two bands still has the topology of a 1-simplex.

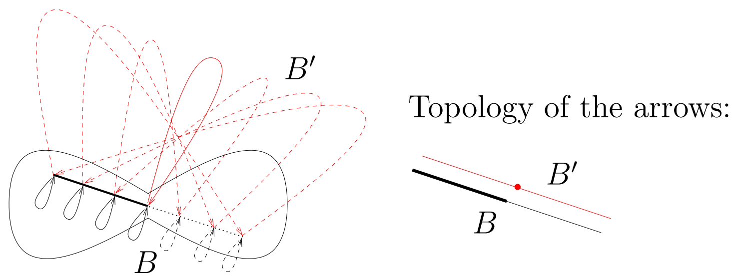



FIGURE 21. Two bands of arrows $B$ and $B'$ in a single chart $\tilde{U}$ with $G_U = C_2$.

At this stage, there is no difference between dashed and continuous lines. This will be important in the next subsection when we start reducing the groupoid by decreasing the set of objects $H_0$.

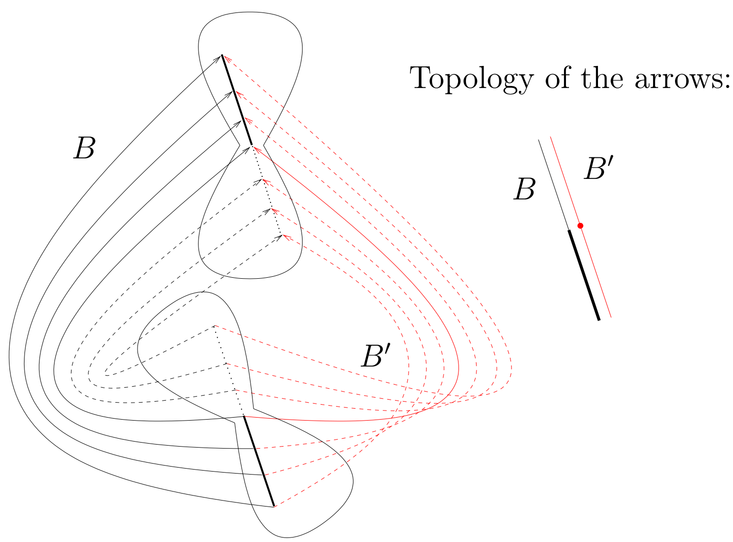



FIGURE 22. Two bands of arrows $B$ and $B'$ from one chart to another.

**Definition 12.2** ($G$-sheaf [**Hae79**])**.** For a topological groupoid $G$ a *$G$-sheaf* is a sheaf of abelian groups on the base $G_0$ together with a continuous (right) action of $G_1$.

The category of all such sheaves, denoted by $\mathrm{Ab}(G)$, has enough injectives.

**Theorem 12.3** ([**MP99**, Theorem 4.1.1])**.** *Any sheaf (of abelian groups) on $\mathcal{M}$ induces an $H$-sheaf, where $H$ is the* HAEFLIGER *groupoid. This defines an equivalence of categories $\mathrm{Ab}(\mathcal{M}) \simeq \mathrm{Ab}(H)$.*

There are several possible constructions of étale proper groupoids $G$ satisfying $\mathrm{Ab}(G) \simeq \mathrm{Ab}(\mathcal{M})$. These groupoids are unique only up to MORITA (=weak) equivalence (cf. [**MP97**]).

**Remark 12.4.** Under such equivalences $\mathrm{Ab}(G) \simeq \mathrm{Ab}(\mathcal{M})$ (as in Theorem 12.3) the global section functor $\Gamma : \mathrm{Ab}(\mathcal{M}) \to \mathrm{Ab}$ corresponds to the functor

$$\Gamma_{\mathrm{inv}} : \mathrm{Ab}(G) \to \mathrm{Ab}$$

of $G$-invariant global sections[14] $\sigma : G_0 \to A$, i.e. $\sigma(y)g = \sigma(x)$ for any arrow $g : y \leftarrow x$.

The cohomology of $G$ with values in $A$ (cf. [**Hae79**]) is the right derived functor cohomology of $\Gamma_{\mathrm{inv}}$

$$H^n(G, A) := (\mathrm{R}^n\, \Gamma_{\mathrm{inv}})(G, A),$$

leading to the natural isomorphism

$$H^n(G, A) \cong H^n(\mathcal{M}, \mathcal{A}). \tag{7}$$

## 13. Reducing the Groupoid

The next step is to replace the HAEFLIGER groupoid by a certain full subgroupoid $R \subset H$ having much less objects but still carrying the same cohomological information.

### 13.1. Retaining the cohomology.

The following Lemma provides a sufficient condition for a subgroupoid $R$ to carry the same cohomological information as the groupoid $G$.

**Lemma 13.1** ([**MP99**, Lemma 4.2.1])**.** *Let $G$ be an étale proper groupoid and $R_0$ a closed subspace of $G_0$. Consider the map*

$$s \circ \pi_2 : R_0 \times_{G_0} G_1 \to G_0, \quad (x, x \xleftarrow{g} y) \mapsto y,$$

*mapping arrows targeting $R_0$ to their source in $G_0$. If $s \circ \pi_2$ is a proper surjection, then the subgroupoid $R \subset G$, defined by the pullback*

$$\begin{tikzcd} R_1 \arrow[r, hook] \arrow[d] & G_1 \arrow[d, "(s,t)"] \\ R_0 \times R_0 \arrow[r, hook] & G_0 \times G_0 \end{tikzcd}$$

*is (by construction) a full subgroupoid and the inclusion $R \subset G$ induces an equivalence of categories $\mathrm{Ab}(R) \simeq \mathrm{Ab}(G)$. Hence, for each $A \in \mathrm{Ab}(G)$ there is a natural isomorphism*

$$H^n(R, A_{|R}) \cong H^n(G, A). \tag{8}$$

Note that $R$ is in general not étale.

[14] Here we confuse $A$ with the étale space of $A$.

13.1.1. *The Reduced Groupoid of a Locally Finite Cover.* Any locally finite cover $\mathcal{F} = \{F_i\}_{i\in I}$ of the coarse space $M$ of the orbifold $\mathcal{M}$ by compact sets, which refines the atlas $\mathcal{U}$, gives rise to such a full subgroupoid $R(\mathcal{F})$ of the HAEFLIGER groupoid $H$ in the following way: Choose for each $F_i$ a chart $U_i \supset F_i$ and a lifting $\tilde{F}_i \subset \tilde{U}_i$, with $\varphi_i : \tilde{U}_i \to U_i$ mapping $\tilde{F}_i$ homeomorphically onto $F_i$. Now define

$$R(\mathcal{F})_0 := \coprod_{i\in I} \tilde{F}_i \subset H_0$$

and take $R(\mathcal{F})_1$ as the pullback of $H_1$ along the inclusion $R(\mathcal{F})_0 \times R(\mathcal{F})_0 \hookrightarrow H_0 \times H_0$ as in Lemma 13.1.

The following Lemma concludes the reduction.

**Lemma 13.2** ([**MP99**, Lemma 4.3.1])**.** *The subgroupoid $R(\mathcal{F})$ constructed above fullfils the condition of Lemma 13.1. Hence there is an equivalence of categories* $\mathrm{Ab}(R(\mathcal{F})) \simeq \mathrm{Ab}(H)$*.*

**13.2. The Reduced Groupoid of a Triangulation.** In this subsection we recall that any triangulation $\mathcal{T}$ of the orbifold $\mathcal{M}$, which is in some sense adapted to the atlas $\mathcal{U}$, induces a locally finite cover $\mathcal{F}_{\mathcal{T}}$, such that the nerve $R(\mathcal{T})_\bullet$ of the reduced groupoid $R(\mathcal{T}) := R(\mathcal{F}_{\mathcal{T}})$ is topologically trivial, i.e. each $R(\mathcal{T})_p$ is a disjoint sum of contractible spaces. This will be further exploited in Section 15.

**Definition 13.3** (Adapted Triangulation)**.** Let $\mathcal{M} = (M, \mathcal{U})$ be an $n$-dimensonal orbifold. A triangulation $\mathcal{T}$ of the coarse topological space $M$ is called adapated to the orbifold atlas $\mathcal{U}$, if

(i) for each simplex $\sigma \in \mathcal{T}$ of maximal dimension $n$, there is a chart $(\tilde{U}_\sigma, G_\sigma, \varphi_\sigma)$ with $\sigma \subset U_\sigma := \varphi_\sigma(\tilde{U}_\sigma)$;

(ii) for each simplex $\tau \in \mathcal{T}$, there is a face $\tau' \subset \tau$, such that the isotropy is constant on $\tau - \tau'$. In particular, each $\tau$ has a (not necessarily unique) vertex $v(\tau)$ with maximal isotropy group, denoted by $G_{v(\tau)}$.

Such a triangulation always exists (cf. [**MP99**, Prop. 1.2.1]).

Now fix a triangulation $\mathcal{T}$ adapted to the atlas $\mathcal{U}$ of $\mathcal{M}$. The simplices of maximal dimension $n$ form a locally finite cover $\mathcal{F}_{\mathcal{T}}$ as in 13.1.1. Hence, as above, we need to fix for each maximal simplex $\sigma$ a chart $\tilde{U}_\sigma$ with $\sigma \subset U_\sigma$ (which exists since $\mathcal{T}$ is adapted to $\mathcal{U}$) and a lifting $\tilde{\sigma} \subset \tilde{U}_\sigma$. With these choices made, define $R(\mathcal{T}) := R(\mathcal{F}_{\mathcal{T}})$. Then, by Lemma 13.2

$$\mathrm{Ab}(R(\mathcal{T})) \cong \mathrm{Ab}(H).$$

In Figures 23 and 24 we indicate the impact a choice of liftings has: The dashed lines and arrows have been deleted from Figures 21 and 22. Only a single arrow survives in the band $B'$.

Figure 25 now sums up the topology of $R(\mathcal{T})_1$ for the $C_2$-teardrop orbifold. The maximal simplices are the unit arrows, and therefore form a copy of $R(\mathcal{T})_0$. The 0- and 1-simplices arise as indicated in Figures 23 and 24. For the topology the "pinheads" do not play a distinguished role. Later, in Section 16, each will serve as the represantative of its connected component. Notice that $R(\mathcal{T})$ is still a proper groupoid, but opposed to $H$ *no longer* étale.

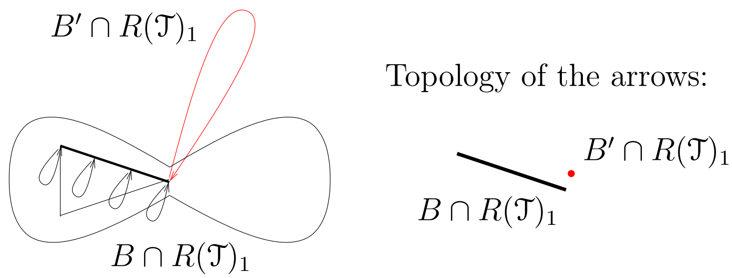



FIGURE 23. Figure 21 after the reduction.

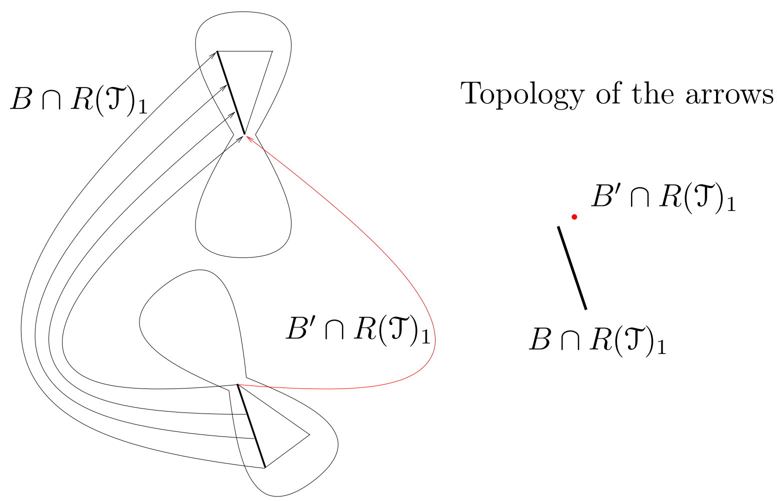



FIGURE 24. Figure 22 after the reduction.

**Proposition 13.4.** *For all integers $p \geq 0$ the space $R(\mathfrak{T})_p$ of the nerve $R(\mathfrak{T})_\bullet$ is homeomorphic to a disjoint sum of simplices.*

PROOF. For $p = 0$ this is true by definition, as $R(\mathfrak{T})_0 := \coprod_{\sigma \in \mathfrak{T}_{\max}} \tilde{\sigma}$. For $p = 1$, the statement is apparent for Figure 25, displaying the topology of the reduced groupoid, and the spirit of the proof is indicated in Figures 23 and 24. A formal and rather technical proof is given in [**MP99**, Proof of Prop. 4.3.3]. The statement for $p > 1$ follows inductively from $p = 0, 1$ [**MP99**, Lemma 4.3.4]. □

Figure 25 only displays the topology of $R(\mathfrak{T})_1$ for the $C_2$-teardrop orbifold. To clarify the multiplicative structure of the groupoid $R(\mathfrak{T})$, Figure 26 shows all possible arrows from $\tilde{a} \to \tilde{b}$, $\tilde{b} \to \tilde{c}$, and $\tilde{a} \to \tilde{c}$ as *continuous* arrows. The dashed arrows are drawn to keep track of the twistings.

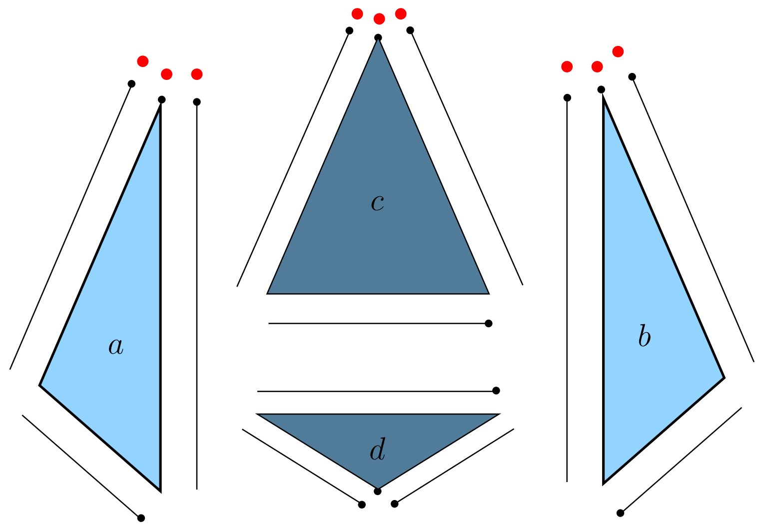



FIGURE 25. The topology of $R(\mathcal{T})_1$

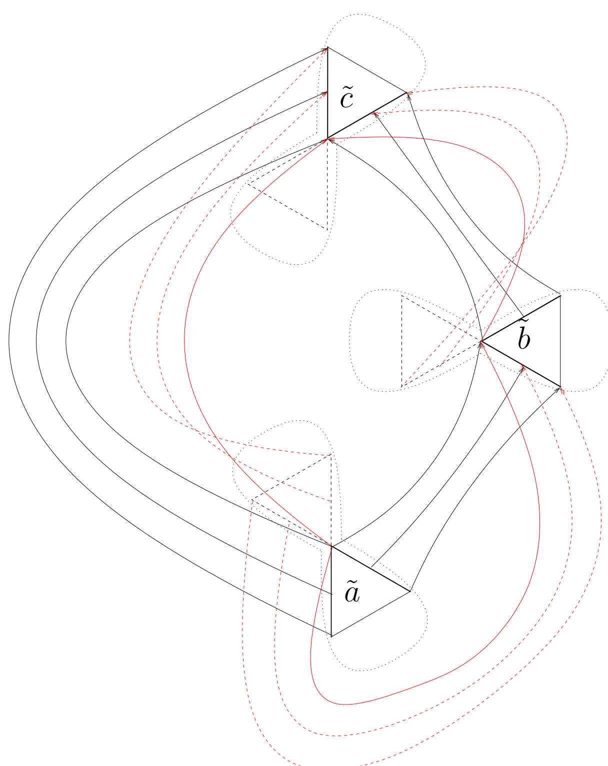



FIGURE 26. The groupoid structure of Figure 25

## 14. From Reduced Groupoid to Simplicial Set

In this section we discuss the simplicial set $\pi_0(R(\mathcal{T})_\bullet)$. The next section shows that this simplicial set is all you need to compute the orbifold cohomology $H^n(\mathcal{M},\mathcal{A})$ with coefficients in a *locally constant* sheaf $\mathcal{A}$.

The following diagram shows the nerve $R(\mathcal{T})_\bullet := \mathrm{Nerve}(R(\mathcal{T}))$ of the reduced groupoid $R(\mathcal{T})$, a simplicial (topological) space with degeneracy maps pointing to the right and face maps pointing left. We further indicate the place of the four structural maps id (unit arrow morphism), $s$ (source), $t$ (target), and $\mu$ (arrow composition):

$$\begin{tikzcd} R(\mathcal{T})_0 \arrow[r, "\mathrm{id}" description] & R(\mathcal{T})_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[r, shift left=1] \arrow[r, shift right=3] & R(\mathcal{T})_2 \arrow[l, shift right=4] \arrow[l, shift left=-1, "\mu" description] \arrow[l, shift left=4] & \cdots \end{tikzcd}$$

All remaining arrows are induced by these four.

Note that $\pi_0$, as a functor from (topological) spaces to sets, induces a functor from simplicial spaces to simplicial sets. Since $R(\mathcal{T})$ is a topological groupoid, its nerve $R(\mathcal{T})_\bullet$ is a simplicial space and $\pi_0(R(\mathcal{T})_\bullet)$ is a simplicial set. For notational simplicity, we set

$$S_\bullet := \pi_0(R(\mathcal{T})_\bullet),$$

with face maps $d_i$ and degeneracy maps $s_i$:

$$\begin{tikzcd} S_0 \arrow[r, "s_0" description] & S_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, shift left=2, "s_0" description] \arrow[r, shift right=2, "s_1" description] & S_2 \arrow[l, shift right=4, "d_0"'] \arrow[l, "d_1" description] \arrow[l, shift left=4, "d_2"] & \cdots \end{tikzcd}$$

In the next section we will state and prove the main result of [**MP99**], reducing the orbifold cohomology of $\mathcal{M}$ with coefficients in a locally constant sheaf $\mathcal{A}$ to a cohomology of the simplicial set $S_\bullet$ with coefficients in an associated *local system* $A$.

## 15. Orbifold Cohomology as Simplicial Cohomology

Let $G$ be an étale topological groupoid. In [**Moe91**, below Thm. 3.1] Moerdijk associates to each $G$-sheaf $A$ a *sheaf* $A^{(\bullet)}$ *over the simplicial space* $G_\bullet := \mathrm{Nerve}(G)$ in the sense of [**Del74**, Def. 5.1.6] (see also [**Tu06**, Cor. 3.8]): For each integer $p \geq 0$ define the *ordinary* sheaf

$$A^{(p)} := \varepsilon_p^*(A)$$

over $G_p := \mathrm{Nerve}(G)_p$ as the pullback along

$$\varepsilon_p : G_p \to G_0,\ g = (x_0 \xleftarrow{g_1} \cdots \xleftarrow{g_p} x_p) \mapsto x_p \tag{9}$$

of the sheaf $A$, viewed as an ordinary sheaf over $G_0$. In particular, for all $g \in G_p$ the stalk

$$A^{(p)}_g = A_{x_p},$$

where $g = (x_0 \xleftarrow{g_1} \cdots \xleftarrow{g_p} x_p)$ (cf. [**MP99**, above Prop. 4.1.2]).

Denote by

$$H^q(G_p, A^{(p)}) = \mathrm{R}^q\,\Gamma(G_p, A^{(p)})$$

the $q$-th (ordinary) sheaf cohomology of $A^{(p)}$. Note that in the definition of $H^q(G_p, A^{(p)})$ all global sections are relevant, while in $H^q(G, A)$ only $G$-invariant global sections are considered. This distinction is especially important for $p = 0$.

The cohomology groups $H^q(G_p, A^{(p)})$ thus form a *cosimplicial abelian group* (with co-face maps and co-degeneracy maps)

$$H^q(G_0, A^{(0)}) \quad \underset{t^*}{\overset{s^*}{\substack{\longrightarrow \\ \xleftarrow{\mathrm{id}^*} \\ \longrightarrow}}} \quad H^q(G_1, A^{(1)}) \quad \substack{\longrightarrow \\ \longleftarrow \\ \xrightarrow{\mu^*} \\ \longleftarrow \\ \longrightarrow} \quad H^q(G_2, A^{(2)}) \quad \cdots ,$$

which we denote[15] by $H^q(G_\bullet, A^{(\bullet)})$, as in [**MP99**, Prop. 4.1.2]. In particular, both global section functors $\Gamma_{\mathrm{inv}}$ and $\Gamma$ are related by

$$\Gamma_{\mathrm{inv}}(G, A) = \ker(\Gamma(G_0, A^{(0)}) \xrightarrow{s^* - t^*} \Gamma(G_1, A^{(1)})). \tag{10}$$

Taking, for fixed $q$, the cohomology of this cosimplicial group yields the *standard spectral sequence of étale topological groupoids*

$$E_2^{p,q} = H^p H^q(G_\bullet, A^{(\bullet)}) \Longrightarrow H^{p+q}(G, A).$$

To explicitly construct the standard spectral sequence one starts with an injective resolution $0 \to A \to I^\bullet$ of $A$ in $\mathrm{Ab}(G)$. This induces injective resolutions $0 \to A^{(q)} \to (I^\bullet)^{(q)}$ of the ordinary sheaves $A^{(q)}$ for all $q \geq 0$. Moerdijk and Pronk then introduce in [**MP99**, Proof of Prop. 4.1.2] the bicomplex[16]

$$B^{p,q} = \Gamma(G_q, (I^p)^{(q)}),$$

where the horizontal maps are induced by the injective resolution $I^\bullet$ and the vertical maps $\Gamma(G_q, (I^p)^{(q)}) \to \Gamma(G_{q+1}, (I^p)^{(q+1)})$ are the alternating sum of the co-face maps of the cosimplicial abelian group $\Gamma(G_\bullet, A^{(\bullet)})$.

By (10), the first spectral sequence ${}^{\mathrm{I}}E$ of the bicomplex collapses (to the $p$-axis) at the first stage giving the row

$$\Gamma_{\mathrm{inv}}(G, I^\bullet) : \Gamma_{\mathrm{inv}}(G, I^0) \to \Gamma_{\mathrm{inv}}(G, I^1) \to \Gamma_{\mathrm{inv}}(G, I^2) \to \cdots ,$$

and hence ${}^{\mathrm{I}}E_2^{n,0} = H^n(G, A)$.

For the second spectral sequence one observes that the $q$-th row of the first sheet ${}^{\mathrm{II}}E_1$ is the cochain complex

$$H^q(G_0, A^{(0)}) \to H^q(G_1, A^{(1)}) \to H^q(G_2, A^{(2)}) \to \cdots .$$

associated to the cosimplicial abelian group $H^q(G_\bullet, A^{(\bullet)})$. Hence ${}^{\mathrm{II}}E_2^{p,q} = H^p H^q(G_\bullet, A^{(\bullet)})$.

**Corollary 15.1** ([**MP99**, Cor. 4.2.2])**.** *For a reduced subgroupoid $R$ as in Lemma 13.1 the standard spectral sequence for $G$ restricts to the spectral sequence*

$$E_2^{p,q} = H^p H^q(R_\bullet, A^{(\bullet)}_{|R_\bullet}) \Longrightarrow H^{p+q}(R, A_{|R}). \tag{11}$$

[15] Deligne uses the same notation in [**Del74**, Def. 5.2.2] to describe what in our context would be $H^q(G, A) := \mathrm{R}^q\, \Gamma_{\mathrm{inv}}(G, A)$.

[16] See [**Del74**, 5.2.3] for the general case.

PROOF. The injective sheaves $(I^p)^{(q)}$ restrict to soft sheaves on the closed subspace $R_p \subset G_p$. Since $R_p$ is paracompact HAUSDORFF space these soft sheaves are $\Gamma$-acyclic and hence compute the sheaf cohomology. $\square$

From now on we assume the sheaf $\mathcal{A} \in \mathrm{Ab}(\mathcal{M})$ to be *locally constant*. The induced sheaves $A^{(p)}_{|R(\mathcal{T})_p}$ are locally constant sheaves on $R(\mathcal{T})_p$, and therefore constant on each connected component. Hence, $A^{(\bullet)}_{|R(\mathcal{T})_\bullet}$ induces a cohomological coefficient system on the simplicial set $S_\bullet = \pi_0(R(\mathcal{T})_\bullet)$ (cf. [**GM03**, I.4.8]), which we also denote by $A$.

**Theorem 15.2** ([**MP99**, Thm. 2.1.1 and Lemma 4.3.5])**.** *Let $\mathcal{M}$ be an orbifold and $\mathcal{A} \in$* Ab($\mathcal{M}$) *a* locally constant *sheaf. Further let $A$ be the local system of coefficients induced by $\mathcal{A}$ on the simplicial set $S_\bullet = \pi_0(R(\mathcal{T})_\bullet)$. Then the orbifold cohomology can be computed via the natural isomorphism*

$$H^p(\mathcal{M}, \mathcal{A}) \cong H^p(S_\bullet, A). \tag{12}$$

PROOF. The above discussion applies for $G := H$, the HAEFLIGER groupoid, with reduced subgroupoid $R := R(\mathcal{T})$. Now, since by Prop. 13.4 each space $R_p$ is a disjoint sum of (contractible) simplices, all higher ($q > 0$) sheaf cohomology groups $H^q(R_p, A^{(p)}_{|R_p})$ vanish. Hence the spectral sequence (11) collapses at the first stage, and in $E_2$ we are left with the row $E_2^{*,0} = H^*(H^0(R_\bullet, A^{(\bullet)}_{|R_\bullet}))$. Furthermore, since $\mathcal{A} \in \mathrm{Ab}(\mathcal{M})$ is assumed locally constant, $H^0(R_\bullet, A^{(\bullet)}_{|R_\bullet})$ can be identified with the cochain complex associated to the simplicial set $S_\bullet = \pi_0(R_\bullet)$ with values in the induced cohomological coefficient system $A$ (cf. [**GM03**, I.4.10 and Formula (I.10)]). In particular $H^p(S_\bullet, A) \cong H^p(H^0(R_\bullet, A^{(\bullet)}_{|R_\bullet}))$, and, summing up, we have the chain of natural isomorphisms

$$H^p(S_\bullet, A) \cong H^p(H^0(R_\bullet, A^{(\bullet)}_{|R_\bullet})) \overset{(11)}{\cong} H^p(R, A_{|R}) \overset{(8)}{\cong} H^p(G, A) \overset{(7)}{\cong} H^p(\mathcal{M}, \mathcal{A}).$$

$\square$

## 16. Describing the Simplicial Set

It is indispensable for explicit computations to describe the elements of $S_\bullet$ in a canonical way. Note that

$$S_0 = \pi_0(R(\mathcal{T})_0) = \pi_0(\coprod_{\sigma \in \mathcal{T}_{\max}} \tilde{\sigma}) = \coprod_{\sigma \in \mathcal{T}_{\max}} \{\tilde{\sigma}\}, \tag{13}$$

which we will denote by $\{\sigma \mid \sigma \in \mathcal{T}_{\max}\}$ following [**MP99**]. This is justified since we fixed a lifting $\tilde{\sigma}$ for each maximal simplex $\sigma$. Now $\sigma$ has two meanings. On the one hand, $\sigma$ is a placeholder for the connected component $\tilde{\sigma}$ of $R(\mathcal{T})_0$, and, on the other hand, it is a subset of the underlying space $M$.

In contrast to this simple description of elements in $S_0$, an element of $S_1$ is a connected component of $R(\mathcal{T})_1$, and elements of $R(\mathcal{T})_1$ are themselves equivalence classes of triples (cf. Definition 12.1).

To describe elements of $S_1$ in a unique way several choices have to be made:

(v) Fix for each simplex $\tau \in \mathfrak{T}$ a *vertex* $v(\tau) \in \mathfrak{T}$ with maximal isotropy (cf. Definition 13.3,(ii));
(c) choose for each non-maximal $\tau \in \mathfrak{T}$ a *chart* $\tilde{U}_\tau$ with $\tau \subset U_\tau$ and
(l) fix a *lifting* $\tilde{\tau} \subset \tilde{U}_\tau$, then
(e) fix *embeddings* $\lambda_{\theta,\tau} : \tilde{U}_\tau \to \tilde{U}_\theta$ for all $\tau \subset \theta$ and $\tau, \theta \in \mathfrak{T}$,

such that

$$\lambda_{\theta,\tau}(\tilde{\tau}) \subset \tilde{\theta},$$

for all $\tau \subset \theta$ and $\tau, \theta \in \mathfrak{T}$.

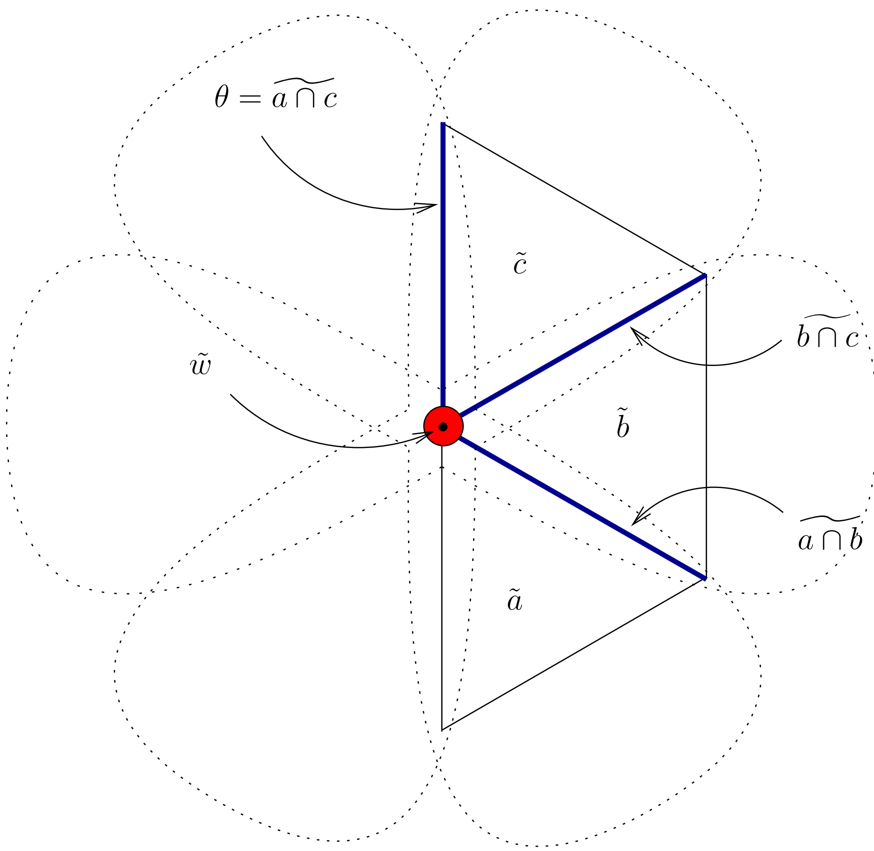



FIGURE 27. Effecting the chosen embeddings in Figure 26.

Let $B$ be a connected component of $R(\mathfrak{T})_1$, i.e. a set of arrows from a maximal simplex $\tilde{\sigma}_1$ to a maximal simplex $\tilde{\sigma}_0$. Note that source $s(B) \subset \tilde{\sigma}_1$ and target $t(B) \subset \tilde{\sigma}_0$, as subsets of $R(\mathfrak{T})_0$, map to the same simplex $\sigma_0 \cap \sigma_1 \subset M$. Let $v := v(\sigma_0 \cap \sigma_1)$ be the chosen vertex in $\sigma_0 \cap \sigma_1$ with maximal isotropy. As indicated by the "pinheads" in Figure 25 we represent $B$ by the *unique* arrow $g \in B$ from $\lambda_{\sigma_1,v}(\tilde{v})$ to $\lambda_{\sigma_0,v}(\tilde{v})$. Thus, $g = [\lambda_1, \tilde{v}, \lambda_0]$, where $\lambda_i$ is an embedding from $\tilde{U}_v$ to $\tilde{U}_{\sigma_i}$.

Since each $\lambda_i$ differs from the fixed embedding $\lambda_{\sigma_i,v}$ only by a unique element $h_i \in G_v$, the arrow $g$ can be written as $g = [\lambda_{\sigma_1,v} \circ h_1, \tilde{v}, \lambda_{\sigma_0,v} \circ h_0]$. Finally, $g_1 := h_1 h_0^{-1}$ is the *unique* element in $G_v$ such that

$$g = [\lambda_{\sigma_1,v} \circ g_1, \tilde{v}, \lambda_{\sigma_0,v}].$$

The element $B \in S_1$ can now be *uniquely* represented by the symbol $(\sigma_0 \xleftarrow{g_1} \sigma_1)$.

Inductively, an element of $S_k = \pi_0(R(\mathfrak{T})_k)$, $k > 0$, can be uniquely represented by a $k$-arrow

$$(\sigma_0 \xleftarrow{g_1} \sigma_1 \leftarrow \cdots \xleftarrow{g_k} \sigma_k),$$

where $\sigma_i \in S_0$, $\sigma_0 \cap \cdots \cap \sigma_k \neq \emptyset$, and $g_i \in G_{v(\sigma_0 \cap \cdots \cap \sigma_k)}$.

Having a unique representation of the elements of $S_\bullet$ we now describe how the arrow composition $\mu$ in $R(\mathfrak{T})$ induces the face maps of $S_\bullet$. Note that the $i$-th face map $d_i : S_k \to S_{k-1}$ simply deletes $\sigma_i$ from a $k$-arrow $(\sigma_0 \xleftarrow{g_1} \sigma_1 \leftarrow \cdots \xleftarrow{g_k} \sigma_k)$. Hence, complying with the way of uniquely representing a $(k-1)$-arrow, we obtain transitions from $G_{v(\sigma_0 \cap \ldots \cap \sigma_k)}$ to $G_{v(\sigma_0 \cap \ldots \cap \widehat{\sigma_i} \cap \ldots \cap \sigma_k)}$. More precisely, for two maximal simplices $\sigma_j, \sigma_l \in \mathfrak{T}$ (or, equivalently, in $S_0$) and for

$$\tau \subset \rho \subset \sigma_j \cap \sigma_l$$

the passage to the quotient $\pi_0(R(\mathfrak{T})_\bullet)$, combined with the choices made above, induces injective maps

$$\nu_{\tau,\rho,\sigma_j,\sigma_l} : G_{v(\tau)} \to G_{v(\rho)}.$$

These are the $\mu$-maps of Moerdijk & Pronk [**MP99**, Subsection 2.2]. We denote them by $\nu$ since we find the name $\mu$ for these maps misleading, as arrow composition in a groupoid (here $R(\mathfrak{T})$) is often denoted by $\mu$. We want to emphasize that the $\nu$'s are simply the *normalizations* discussed above, *completely independent* of the arrow composition in the groupoid. The only leftover of the arrow composition of the groupoid $R(\mathfrak{T})$ are the multiplication laws of the different isotropy groups.

With the normalization $\nu$ at hand, the face maps $d_j$ can be explicitly described by

$$d_j(\sigma_0 \xleftarrow{g_1} \cdots \xleftarrow{g_k} \sigma_k) = \begin{cases} \sigma_1 \xleftarrow{\nu(g_2)} \cdots \xleftarrow{\nu(g_k)} \sigma_k, & j = 0 \\ \sigma_0 \xleftarrow{\nu(g_1)} \cdots \leftarrow \sigma_{j-1} \xleftarrow{\nu(g_j g_{j+1})} \sigma_{j+1} \leftarrow \cdots \xleftarrow{\nu(g_k)} \sigma_k, & 0 < j < k \\ \sigma_0 \xleftarrow{\nu(g_1)} \cdots \xleftarrow{\nu(g_{k-1})} \sigma_{k-1}, & j = k, \end{cases}$$

where the different $\nu$'s are defined as follows: For $\tau := \sigma_0 \cap \cdots \cap \sigma_k$ and $\rho := \sigma_0 \cap \cdots \widehat{\sigma_j} \cdots \cap \sigma_k$

$$\nu(g_j) = \nu_{\tau,\rho,\sigma_{j-1},\sigma_j}(g_j) \quad \text{and} \quad \nu(g_j g_{j+1}) = \nu_{\tau,\rho,\sigma_{j-1}\sigma_{j+1}}(g_j g_{j+1}).$$

It is noteworthy that, in general, $\nu(1_{G_{v(\tau)}}) \neq 1_{G_{v(\rho)}}$, however, the identity

$$\nu(g_j g_{j+1}) = \nu(g_j)\nu(g_{j+1})$$

holds.

## 17. Examples

**Example 17.1** (Cohomology of a finite group)**.** Let $G$ be a finite subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and $\mathcal{M} := \mathbb{B}^n/G$ the quotient orbifold, where $\mathbb{B}^n$ is the closed unit $n$-ball. A locally constant sheaf $\mathcal{A}$ on $\mathcal{M}$ is uniquely determined by a $G$-module $A$, where $A = \mathcal{A}_0$ is the stalk of $\mathcal{A}$ at the origin of $\mathbb{R}^n$. Then the orbifold cohomology of $\mathcal{M}$ coincides with the group cohomology of $G$: $H^n(\mathcal{M}, \mathcal{A}) \cong H^n(G, A)$. However, computing the group cohomology in such a way is highly ineffective, since $|S_0| > 1$.

**Example 17.2** (SCO package computations)**.** We consider the global quotient orbifolds $\mathcal{M} := \mathbb{R}^2/R$, where $R$ is one of the 17 two dimensional space groups. For the triangulation of the respective fundamental domains see [Gö08].

| Group | $H^1$ | $H^2$ | $H^3$ | $H^4$ | $H^5$ | up to dim. |
|---|---|---|---|---|---|---|
| p1 | $\mathbb{Z}^2$ | $\mathbb{Z}$ | 0 | 0 | 0 | - |
| p2 | 0 | $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$ | 0 | $(\mathbb{Z}/2\mathbb{Z})^4$ | 0 | $> 10$ |
| p3 | 0 | $\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2$ | 0 | $(\mathbb{Z}/3\mathbb{Z})^3$ | 0 | 7 |
| p4 | 0 | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ | 0 | $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2$ | 0 | 5 |
| p6 | 0 | $\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ | 0 | $(\mathbb{Z}/6\mathbb{Z})^2$ | ?[1] | 4, 5[2] |
| pm | $\mathbb{Z}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $> 10$ |
| pg | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0 | 0 | 0 | - |
| cm | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $> 10$ |
| pmm | 0 | $(\mathbb{Z}/2\mathbb{Z})^4$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | $(\mathbb{Z}/2\mathbb{Z})^8$ | $(\mathbb{Z}/2\mathbb{Z})^8$ | 8 |
| cmm | 0 | $(\mathbb{Z}/2\mathbb{Z})^3$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^5$ | $(\mathbb{Z}/2\mathbb{Z})^4$ | 8 |
| pmg | 0 | $(\mathbb{Z}/2\mathbb{Z})^3$ | $\mathbb{Z}/2\mathbb{Z}$ | $(\mathbb{Z}/2\mathbb{Z})^3$ | $\mathbb{Z}/2\mathbb{Z}$ | $> 10$ |
| pgg | 0 | $(\mathbb{Z}/2\mathbb{Z})^3$ | 0 | $(\mathbb{Z}/2\mathbb{Z})^2$ | 0 | $> 10$ |
| p4m | 0 | $(\mathbb{Z}/2\mathbb{Z})^3$ | $(\mathbb{Z}/2\mathbb{Z})^3$ | $(\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4\mathbb{Z})^2$ | ? | 4, 5[2] |
| p4g | 0 | $(\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z}$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}/4\mathbb{Z}$ | $(\mathbb{Z}/2\mathbb{Z})^3$ | 5 |
| p3m1 | 0 | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^3$ | $\mathbb{Z}/2\mathbb{Z}$ | 5 |
| p31m | 0 | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2$ | $\mathbb{Z}/2\mathbb{Z}$ | 5 |
| p6m | 0 | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | $(\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/3\mathbb{Z})^2$ | ? | 4, 5[2] |

Note that $H^0 = \mathbb{Z}$ in all cases.

**Lemma 17.3.** *First, we state some well-known group cohomologies.*
*The cohomology of* $\mathrm{p1} = \mathbb{Z}^2$ *is the cohomology of the Torus:*

$$H^i(\mathrm{p1}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i \geq 3 \end{cases}$$

[1] obtained easily with Mayer-Vietoris or Lyndon/Hochschild-Serre.

[2] when computing over $\mathbb{F}_2$, losing $\mathbb{Z}/3\mathbb{Z}$-torsion.

*Similarly, the cohomology of* pg *is the cohomology of the famous Klein Bottle:*

$$H^i(\mathrm{pg},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z}/2\mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases}$$

*The cohomology of $C_n$ for $n \geq 2$ is*

$$H^i(C_n,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i \text{ odd} \\ \mathbb{Z}/n\mathbb{Z} & i\neq 0 \text{ even} \end{cases}$$

*For the nontrivial action of $C_2$ on $\mathbb{Z}$, we get*

$$H^i(C_2,\mathbb{Z}_{(-1)}) = \begin{cases} 0 & i \text{ even} \\ \mathbb{Z}/2\mathbb{Z} & i \text{ odd} \end{cases}$$

*Cohomologies over both $\mathbb{Z}$ and $\mathbb{Z}^2$ as nontrivial modules are important for the top two rows of the* Lyndon/Hochschild-Serre *spectral sequence, but in these examples usually easily computed.*

Now we start with those wallpaper groups whose cohomology is attainable through theoretical methods. Our main tool will be the Lyndon/Hochschild-Serre spectral sequence, as we always have a $\mathbb{Z}^2$-factor in each wallpaper group. The three rows of the aforementioned spectral sequence are the cohomologies of the factor groups with values in $\mathbb{Z}$, $\mathbb{Z}^2$, and $\mathbb{Z}$ - the latter two each with a possibly nontrivial $R/\mathbb{Z}^2$-action, where $R$ denotes the space group. To support the fact that each row corresponds to a finite group cohomology with values in $H^q(\mathbb{Z}^2,\mathbb{Z})$, $q=0,1,2$, we index the rows by $q$, such that the bottom row is the 0-th. This row is always the $R/\mathbb{Z}^2$-cohomology with values in the trivial module $\mathbb{Z}$.

**Theorem 17.4.** *The cohomology of* p2 *is*

$$H^i(\mathrm{p2},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}\oplus(\mathbb{Z}/2)^3 & i=2 \\ 0 & i\geq 3 \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^4 & i\geq 4 \text{ even} \end{cases}$$

Proof. Note that $\mathrm{p2} = C_2 \ltimes \mathbb{Z}^2$, where $C_2$ acts on $\mathbb{Z}^2$ via $\alpha \mapsto \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. An easy calculation with this operation yields the following Lyndon/Hochschild-Serre spectral sequence. All morphisms are zero, as this is $E_2^{p,q}$.

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \cdots \\ 0 & (\mathbb{Z}/2\mathbb{Z})^2 & 0 & (\mathbb{Z}/2\mathbb{Z})^2 & 0 & (\mathbb{Z}/2\mathbb{Z})^2 & \cdots \\ \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \cdots \end{array}$$

Adding up the diagonals provides the p2 cohomology. There is a small extension problem at $H^2$, which can be solved in different ways. We showcase one.

$$\mathrm{t}(H^2(R,\mathbb{Z})) \cong \mathrm{t}(H_1(R,\mathbb{Z})) = \mathrm{t}(R/R'),$$
$$R/R' \cong P/P' \times \mathbb{Z}^2/[P,\mathbb{Z}^2]$$

Here $P = C_2$ is abelian, therefore $P/P' = C_2 \cong \mathbb{Z}/2\mathbb{Z}$. On the other hand, $[P,\mathbb{Z}^2]$ is generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

hence the group of coinvariants $\mathbb{Z}^2/[P,\mathbb{Z}^2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Combined we obtain

$$\mathrm{t}(H^2(R,\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 \cong (\mathbb{Z}/2\mathbb{Z})^3,$$

where the first summand resides in the 0-th row of the spectral sequence, and the second one in the first row.

We omit this proof in the following theorems. Note that this result and more (in fact, the whole group cohomology over $\mathbb{Z}$ up to a certain degree) can also be obtained by computations with the SCO package [Gör08]. $\square$

**Theorem 17.5.** *The cohomology of* p3 *is*

$$H^i(\mathrm{p3},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z} \oplus (\mathbb{Z}/3)^2 & i = 2 \\ 0 & i \geq 3 \text{ odd} \\ (\mathbb{Z}/3\mathbb{Z})^3 & i \geq 4 \text{ even} \end{cases}$$

Proof. Note that $\mathrm{p3} = C_3 \ltimes \mathbb{Z}^2$, where $C_3$ acts on $\mathbb{Z}^2$ via $\alpha \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. As above we get a spectral sequence, where all morphisms at this stage are zero.

$$\begin{matrix} \mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & \cdots \\ 0 & \mathbb{Z}/3\mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & \cdots \\ \mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & \mathbb{Z}/3\mathbb{Z} & 0 & \cdots \end{matrix}$$

Adding up the diagonals provides the p3 cohomology. $\square$

**Theorem 17.6.** *The cohomology of* p4 *is*

$$H^i(\mathrm{p4},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} & i = 2 \\ 0 & i \geq 3 \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2 & i \geq 4 \text{ even} \end{cases}$$

PROOF. Note that p4 $= C_4 \ltimes \mathbb{Z}^2$, where $C_4$ acts on $\mathbb{Z}^2$ via $\alpha \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The same procedure as above yields

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \cdots \\ 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \cdots \\ \mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \mathbb{Z}/4\mathbb{Z} & 0 & \cdots \end{array}$$

Adding up the diagonals provides the p4 cohomology. □

**Theorem 17.7.** *The cohomology of* p6 *is*

$$H^i(\mathrm{p6}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & i = 2 \\ 0 & i \geq 3 \text{ odd} \\ (\mathbb{Z}/6\mathbb{Z})^2 & i \geq 4 \text{ even} \end{cases}$$

PROOF. Note that p6 $= C_6 \ltimes \mathbb{Z}^2$, where $C_6$ acts on $\mathbb{Z}^2$ via $\alpha \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. The same procedure as above yields

$$\begin{array}{ccccccc} \mathbb{Z} & 0 & \mathbb{Z}/6\mathbb{Z} & 0 & \mathbb{Z}/6\mathbb{Z} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mathbb{Z} & 0 & \mathbb{Z}/6\mathbb{Z} & 0 & \mathbb{Z}/6\mathbb{Z} & 0 & \cdots \end{array}$$

Adding up the diagonals provides the p6 cohomology. □

We have seen that some cohomologies of infinite groups can be calculated by analyzing the LYNDON/HOCHSCHILD-SERRE spectral sequence and taking care of extension problems. However, as soon as the groups become more complicated, so do these problems. We are aware of the fact that there is a plethora of other tricks and computational methods to obtain group cohomologies. One of these is the Perturbation Lemma by C.T.C WALL, which is used to great effect by the `HAP` project [**HAP08**].

However, the computation of group cohomology is not our main concern. In fact, SCO enables us to compute general orbifold cohomology as defined in [**MP99**]. The 17 wallpaper groups and their operation on $\mathbb{R}^2$ are aesthetically pleasing examples of orbifolds that happen to be induced by infinite groups, showcasing the fact that orbifold cohomology does in some cases generalize group cohomology. It should also be noted that the orbifolds we obtain are of finite dimension and there seems to be some hidden relatedness between the look of these orbifolds and the cohomology one obtains.

Take, for example, the orbifolds corresponding to p2 and p6, respectively:

Taking into account the fact that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and comparing these pictures to the cohomology above, it is clear to see that the group cohomology of p2 and p6 is quite related to the orbifold representation of their fundamental domains. This is just one of many examples of this nature.

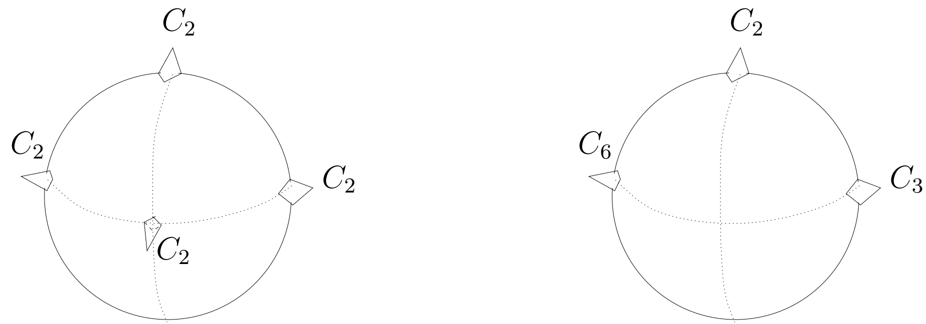



FIGURE 28. Comparing the orbifolds of p2 and p6

**Conjecture 17.8.** *The integral group cohomology of the remaining wallpaper groups is as follows:*

| | | |
|---|---|---|
| $H^i(\mathrm{pm}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^2 & i \geq 2 \end{cases}$ |
| $H^i(\mathrm{cm}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ \mathbb{Z}/2\mathbb{Z} & i \geq 2 \end{cases}$ |
| $H^i(\mathrm{pmm}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^{4\lfloor \frac{i}{2} \rfloor} & i \geq 2 \end{cases}$ |
| $H^i(\mathrm{cmm}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^{i+1} & i \geq 2 \textit{ even} \\ (\mathbb{Z}/2\mathbb{Z})^{i-1} & i \geq 3 \textit{ odd} \end{cases}$ |
| $H^i(\mathrm{pmg}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^3 & i \geq 2 \textit{ even} \\ \mathbb{Z}/2\mathbb{Z} & i \geq 3 \textit{ odd} \end{cases}$ |
| $H^i(\mathrm{pgg}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 2 \\ 0 & i \geq 3 \textit{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^2 & i \geq 4 \textit{ even} \end{cases}$ |
| $H^i(\mathrm{p4m}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^3 & i = 2, 3 \\ (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4\mathbb{Z})^2 & i = 4 \\ \textit{unknown}[1] & i \geq 5 \end{cases}$ |
| $H^i(\mathrm{p4g}, \mathbb{Z})$ | $=$ | $\begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^{\frac{i}{2}+1} \oplus \mathbb{Z}/4\mathbb{Z} & i \geq 2 \textit{ even} \\ (\mathbb{Z}/2\mathbb{Z})^{\frac{i+1}{2}} & i \geq 3 \textit{ odd} \end{cases}$ |

$$H^i(\mathrm{p3m1}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}/2\mathbb{Z} & i \geq 2,\ i \not\equiv_4 0 \\ \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^3 & i \geq 4,\ i \equiv_4 0 \end{cases}$$

$$H^i(\mathrm{p31m}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & i \geq 2,\ i \equiv_4 2 \\ \mathbb{Z}/2\mathbb{Z} & i \geq 3,\ i \equiv_4 1, 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2 & i \geq 4,\ i \equiv_4 0 \end{cases}$$

$$H^i(\mathrm{p6m}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ (\mathbb{Z}/2\mathbb{Z})^2 & i = 2, 3 \\ (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/3\mathbb{Z})^2 & i = 4 \\ \textit{unknown}^{1} & i \geq 5 \end{cases}$$

[1] and, of course, nonperiodic.

# Bibliography


[AB69] Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969. MR MR0269685 (42 #4580) 9, 39

[Bar09] Mohamed Barakat, *The* `homalg` *package* – GAP4, 2007-2009, (`http://homalg.math.rwth-aachen.de/index.php/core-packages/homalg-package`). 3, 10, 31, 68

[BB] Mohamed Barakat and Barbara Bremer, *Higher Extension Modules and the Yoneda Product*, submitted (`arXiv:math.KT/0802.3179`). 68

[Bjö79] J.-E. Björk, *Rings of differential operators*, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam, 1979. MR MR549189 (82g:32013) 9, 37, 38, 39, 40

[BR08] Mohamed Barakat and Daniel Robertz, `homalg` – *A Meta-Package for Homological Algebra*, J. Algebra Appl. **7** (2008), no. 3, 299–317, (`arXiv:math.AC/0701146`). MR MR2431811 24, 37, 41, 67, 68

[Con94] Alain Connes, *Noncommutative geometry*, Academic Press Inc., San Diego, CA, 1994, (`http://www.alainconnes.org/downloads.html`). MR 95j:46063 45

[CQR05] F. Chyzak, A. Quadrat, and D. Robertz, *Effective algorithms for parametrizing linear control systems over Ore algebras*, Appl. Algebra Engrg. Comm. Comput. **16** (2005), no. 5, 319–376, (`http://www-sop.inria.fr/members/Alban.Quadrat/PubsTemporaire/AAECC.pdf`). MR MR2233761 (2007c:93041) 39

[Del74] Pierre Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77. MR MR0498552 (58 #16653b) 51, 52

[GÖ8] Simon Görtzen, *Simplicial cohomology of orbifolds*, Master's thesis, RWTH Aachen University, 2008, `http://wwwb.math.rwth-aachen.de/goertzen/`. 43, 56

[GM03] Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, 2. ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR MR1950475 (2003m:18001) 53

[Gör08] S. Görtzen, *Gap package sco*, 2007-2008, `http://wwwb.math.rwth-aachen.de/goertzen/SCO`. 43, 58

[GP02] Gert-Martin Greuel and Gerhard Pfister, *A* **singular** *introduction to commutative algebra*, Springer-Verlag, Berlin, 2002, With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh, and UNIX). MR MR1930604 (2003k:13001) 41

[Hae79] A. Haefliger, *Differential cohomology*, Differential topology (Varenna, 1976), Liguori, Naples, 1979, pp. 19–70. MR MR660658 (83k:17014) 46, 47

[Hae84] André Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), no. 116, 70–97, Transversal structure of foliations (Toulouse, 1982). MR MR755163 (86c:57026a) 45

[HAP08] *HAP – Homological Algebra Programming*, 2008, (`http://hamilton.nuigalway.ie/Hap/www/`). 59

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116) 11

[HL97] Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997. MR MR1450870 (98g:14012) 9, 37

[HS97] P. J. Hilton and U. Stammbach, *A course in homological algebra*, second ed., Graduate Texts in Mathematics, vol. 4, Springer-Verlag, New York, 1997. MR MR1438546 (97k:18001) 34, 39

[ht09] The homalg team, *The* `homalg` *project*, 2003-2009, (`http://homalg.math.rwth-aachen.de/`). 3, 69

[Kas95] Masaki Kashiwara, *Algebraic study of systems of partial differential equations*, Mém. Soc. Math. France (N.S.) (1995), no. 63, xiv+72. MR MR1384226 (97f:32012) 9

[Moe91] Ieke Moerdijk, *Classifying toposes and foliations*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 1, 189–209. MR MR1112197 (92i:57028) 51

[MP97] I. Moerdijk and D. A. Pronk, *Orbifolds, sheaves and groupoids*, $K$-Theory **12** (1997), no. 1, 3–21. MR MR1466622 (98i:22004) 46

[MP99] ———, *Simplicial cohomology of orbifolds*, Indag. Math. (N.S.) **10** (1999), no. 2, 269–293, (`arXiv:q-alg/9708021`). MR MR1816220 (2002b:55012) 43, 44, 46, 47, 48, 49, 51, 52, 53, 55, 59

[MR01] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, revised ed., Graduate Studies in Mathematics, vol. 30, American Mathematical Society, Providence, RI, 2001, With the cooperation of L. W. Small. MR MR1811901 (2001i:16039) 36

[Noe27] Emmy Noether, *Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern*, Math. Ann. **96** (1927), no. 1, 26–61. MR MR1512304 11

[OT01] Toshinori Oaku and Nobuki Takayama, *Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups*, J. Pure Appl. Algebra **156** (2001), no. 2-3, 267–308. MR MR1808827 (2002i:13036) 41

[PQ00] J. F. Pommaret and A. Quadrat, *Equivalences of linear control systems*, Proc. Int. Symp. Mathematical Theory of Networks and Systems, MTNS 2000, Perpignan, France, 2000, (`http://www-sop.inria.fr/members/Alban.Quadrat/Publications/MTNS.pdf`). 39

[Pro95] Dorette A. Pronk, *Groupoid representations for sheaves on orbifolds*, Ph.D. thesis, Department of Mathematics, Utrecht University, 1995. 45

[Qua99] Alban Quadrat, *Analyse algébrique des systèmes de contrôle linéaires multidimensionnels*, Ph.D. thesis, Ecole Nationale des Ponts et Chaussées, CERMICS, France, September 1999. 39

[Qua01] ———, *Extended Bézout Identities*, Proceedings of the European Control Conference, 2001, (`http://www-sop.inria.fr/members/Alban.Quadrat/Publications/ECC.pdf`). 9, 38, 39

[Roo62] Jan-Erik Roos, *Bidualité et structure des foncteurs dérivés de* $\varinjlim$ *dans la catégorie des modules sur un anneau régulier*, C. R. Acad. Sci. Paris **254** (1962), 1556–1558. MR MR0136639 (25 #106a) 9, 37

[Rot79] Joseph J. Rotman, *An introduction to homological algebra*, Pure and Applied Mathematics, vol. 85, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979. MR MR538169 (80k:18001) 7, 35

[RS02] Julio Rubio and Francis Sergeraert, *Constructive algebraic topology*, Bull. Sci. Math. **126** (2002), no. 5, 389–412, (`arXiv:math.AT/0111243`). MR MR1914728 (2003g:55001) 10

[RSS] Julio Rubio, Francis Sergeraert, and Yvon Siret, *Kenzo — a Symbolic Software for Effective Homology Computation*, (`http://www-fourier.ujf-grenoble.fr/~sergerar/`). 10

[Tu06] Jean-Louis Tu, *Groupoid cohomology and extensions*, Trans. Amer. Math. Soc. **358** (2006), no. 11, 4721–4747 (electronic). MR MR2231869 (2007i:22008) 51

[TW01] Harrison Tsai and Uli Walther, *Computing homomorphisms between holonomic $D$-modules*, J. Symbolic Comput. **32** (2001), no. 6, 597–617, Effective methods in rings of differential operators. MR MR1866706 (2002k:16051) 40, 41

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR MR1269324 (95f:18001) 7, 10, 15, 25, 26, 27, 28, 29, 34

# Appendices

## A. The triangulation algorithm

**Definition A.1** (Computable ring [**BR08**, Subsection 1.2])**.** A left and right noetherian ring is called **computable** if there exists an algorithm which solves one sided inhomogeneous linear systems $XA = B$ and $AX = B$, where $A$ and $B$ are matrices with entries in $D$. The word "solves" means: The algorithm can decide if a solution exists, and, if solvable, is able to compute a particular solution as well as a finite generating set of solutions of the corresponding homogeneous system.

From now on the ring $D$ is assumed computable. Let $M$ be a finitely generated left $D$-module. Then $M$ is finitely presented, i.e. there exists a matrix $\mathtt{M} \in D^{p\times q}$, viewed as a morphism $\mathtt{M} : D^{1\times p} \to D^{1\times q}$, such that $\operatorname{coker} \mathtt{M} \cong M$. $\mathtt{M}$ is called a **matrix of relations** or a **presentation matrix** for $M$. It forms the beginning of a free resolution

$$0 \leftarrow M \leftarrow D^{1\times q} \xleftarrow{d_1=\mathtt{M}} D^{1\times p} \xleftarrow{d_2} D^{1\times p_2} \xleftarrow{d_3} \cdots .$$

$d_i$ is called the $i$-th syzygies matrix of $M$ and $K_i := \operatorname{coker} d_{i+1}$ the $i$-th syzygies module. $K_i$ is uniquely determined by $M$ up to **projective equivalence**.

Now suppose that $M$ is endowed with an $m$-filtration $F = (F_pM)$. We will sketch an algorithm that, starting from a presentation matrix $\mathtt{M} \in D^{p\times q}$ for $M$ and presentation matrices $\mathtt{M}_p$ for the graded parts $M_p := \operatorname{gr}_p M$, computes another **upper triangular** presentation matrix $\mathtt{M}_F$ of the form[17]

$$\mathtt{M}_F = \begin{pmatrix} \mathtt{M}_{p_{m-1}} & * & \cdots & \cdots & * \\ & \mathtt{M}_{p_{m-2}} & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & \mathtt{M}_{p_1} & * \\ & & & & \mathtt{M}_{p_0} \end{pmatrix} \in D^{p'\times q'}$$

and an isomorphism $\operatorname{coker} \mathtt{M}_F \xrightarrow{\cong} \operatorname{coker} \mathtt{M}$ given by a matrix $\mathtt{T} \in D^{q'\times q}$:

Let $(\psi_p)$ be an ascending $m$-filtration system computing $F$ (cf. Def. 4.3). To start the induction take $p$ to be the highest degree $p_{m-1}$ in the filtration and set $\mathtt{F}_p\mathtt{M} := \mathtt{M}$. Since

$$\mu_p := \psi_p : M_p = \operatorname{coker} \mathtt{M}_p \to \operatorname{coker} \mathtt{F}_p\mathtt{M}$$

[17] Note that choosing a generating system of $M$ adapted to the filtration $F$ is not enough to produce a triangular presentation matrix, as changing the set of generators only corresponds to column operations on $\mathtt{M}$.

is a generalized isomorphism, its unique generalized inverse exists and is an epimorphism (cf. Cor. 4.8), which we denote by $\pi_p : F_pM \to M_p$. (Note: $\operatorname{coker} \mathtt{F}_p\mathtt{M} = F_pM = M$ for $p = p_{m-1}$.) Since $D$ is computable we are able to determine (a matrix of) an injective morphism $\iota_p$ mapping onto the kernel of $\pi_p$. The source of $\iota_p$ is a module isomorphic to $F_{p-1}M$, which we also denote by $F_{p-1}M$. No confusion can occur since we will only refer to the latter. All maps in the exact-rows diagram

$$\begin{tikzcd}
0 & M_p \arrow[l] \arrow[d, equal] & P_0 \arrow[l, "\nu"'] \arrow[d, "\eta_0"] & K_1 \arrow[l, "\mathtt{M}_p"'] \arrow[d, "\eta"] & 0 \arrow[l] \\
0 & M_p \arrow[l] & F_pM \arrow[l, "\pi_p"'] & F_{p-1}M \arrow[l, "\iota_p"'] & 0 \arrow[l]
\end{tikzcd}$$

are computable, where $P_0$ is a free $D$-module and $K_1$ is the 1-st syzygies module of $\mathtt{M}_p$: $\eta_0$ is computable since $P_0$ is free and $\eta$ is computable since $\iota_p$ is injective (see [BR08, Subsection 3.1]). This yields the short exact sequence

$$0 \to K_1 \xrightarrow{\kappa:=\begin{pmatrix}\mathtt{M}_p & \eta\end{pmatrix}} P_0 \oplus F_{p-1}M \xrightarrow{\rho:=\begin{pmatrix}-\eta_0 \\ \iota_p\end{pmatrix}} F_pM \to 0.$$

Hence, the cokernel of $\kappa := \begin{pmatrix}\mathtt{M}_p & \eta\end{pmatrix}$ is isomorphic to $F_pM$ which therefore admits a presentation matrix of the form

$$\mathtt{M}_F^p = \begin{pmatrix}\mathtt{M}_p & \eta \\ 0 & \mathtt{F}_{p-1}\mathtt{M}\end{pmatrix},$$

where $\mathtt{F}_{p-1}\mathtt{M}$ is a presentation matrix of $F_{p-1}M$ (for more details see [BB, Subsection 7.1]). If $\chi : P_0 \oplus F_{p-1}M \to \operatorname{coker} \kappa = \operatorname{coker} \mathtt{M}_F^p$ denotes the natural epimorphism and $\rho := \begin{pmatrix}-\eta_0 \\ \iota_p\end{pmatrix}$, then the matrix $\mathtt{T}^p$ of the morphism $T^p := \rho \circ \chi^{-1}$ is an isomorphism between $\operatorname{coker} \mathtt{M}_F^p$ and $\operatorname{coker} \mathtt{F}_p\mathtt{M}$. By the induction hypothesis we have

$$\widetilde{\mathtt{M}}_F^{p+1} := \left(\begin{array}{c|c}\text{stable}_p & \eta_p \\ \hline 0 & \mathtt{F}_p\mathtt{M}\end{array}\right) = \left(\begin{array}{c|c|c}\text{stable}_{p+1} & * & * \\ \hline 0 & M_{p+1} & * \\ \hline 0 & 0 & \mathtt{F}_p\mathtt{M}\end{array}\right) = \left(\begin{array}{c|c}\text{stable}_{p+1} & * \, * \\ \hline 0 & \mathtt{M}_F^{p+1}\end{array}\right)$$

with $\operatorname{coker} \widetilde{\mathtt{M}}_F^{p+1} \cong \operatorname{coker} \mathtt{M}$. (Since $p$ was decreased by one the old $\mathtt{F}_{p-1}\mathtt{M}$ is now addressed as $\mathtt{F}_p\mathtt{M}$, etc.). Before proceeding inductively on the submatrix $\mathtt{F}_p\mathtt{M}$ of $\widetilde{\mathtt{M}}_F^{p+1}$ take the quotient

$$\mu_p := (\iota_{p_{m-1}} \circ \cdots \circ \iota_{p+1})^{-1} \circ \psi_p : M_p = \operatorname{coker} \mathtt{M}_p \to \operatorname{coker} \mathtt{F}_p\mathtt{M},$$

which is like $\mu_{p+1}$ again a generalized isomorphism. Note that matrix $\mathtt{T}^p$ of the morphism $T^p := \rho \circ \chi^{-1}$ providing the isomorphism between $\operatorname{coker} \mathtt{M}_F^p$ and $\operatorname{coker} \mathtt{F}_p\mathtt{M}$ now has to be multiplied from the right to the submatrix $\eta_p$ of $\widetilde{\mathtt{M}}_F^{p+1}$ which lies above $\mathtt{F}_p\mathtt{M}$. This completes the induction. The algorithm terminates with $\mathtt{M}_F := \widetilde{\mathtt{M}}_F^{p_0}$ and $\mathtt{T}$ is the composition of all the successive column operations on $\mathtt{M}$. $\square$

The above algorithm is implemented in `homalg` package [Bar09] under the name `IsomorphismOfFiltration`. It takes an $m$-filtration system $(\psi_p)$ of $M = \operatorname{coker} \mathtt{M}$ as its input and returns an isomorphism $\tau : \operatorname{coker} \mathtt{M}_F \to \operatorname{coker} \mathtt{M}$ with a triangular presentation matrix $\mathtt{M}_F$, as described above. `IsomorphismOfFiltration` will be extensively used in the examples in Appendix B.

## B. Examples with GAP's homalg

The packages `homalg`, `IO_ForHomalg`, and `RingsForHomalg` are assumed loaded:

```
gap> LoadPackage( "RingsForHomalg" );
true
```

For details see the `homalg` project [ht09].

**Example B.1** (`LeftPresentation`)**.** Define a left module $W$ over the polynomial ring $D := \mathbb{Q}[x, y, z]$. Also define its right mirror $Y$.

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> wmat := HomalgMatrix( "[ \
x*y,  y*z,    z,        0,         0,    \
x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
x^4,  x^3*z,  0,        x^2*z,     -x*z, \
0,    0,      x*y,      -y^2,      x^2-1,\
0,    0,      x^2*z,    -x*y*z,    y*z,  \
0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
]", 6, 5, Qxyz );
<A homalg external 6 by 5 matrix>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> Y := Hom( Qxyz, W );
<A right module on 5 generators satisfying 6 relations>
```

**Example B.2** (Homological `GrothendieckSpectralSequence`)**.** Example B.1 continued. Compute the double-Ext spectral sequence for $F := \operatorname{Hom}(-, Y)$, $G := \operatorname{Hom}(-, D)$, and the $D$-module $W$. This is an example for Subsection 9.1.1.

```
gap> F := InsertObjectInMultiFunctor( Functor_Hom, 2, Y, "TensorY" );
<The functor TensorY>
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable homological spectral sequence with sheets at levels [ 0 .. 4 ]
each consisting of left modules at bidegrees [ -3 .. 0 ]x[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
```

```
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
```

```
---------
Level 3:

 * s s s
 * s s s
 . . s *
 . . . *
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:
   0: <A non-zero left module presented by 33 relations for 23 generators>
  -1: <A non-zero left module presented by 37 relations for 22 generators>
  -2: <A non-zero left module presented by 20 relations for 8 generators>
  -3: <A non-zero left module presented by 29 relations for 4 generators>
of
<A non-zero left module presented by 111 relations for 37 generators>>
gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:
   0: <A non-zero left module presented by 25 relations for 16 generators>
  -1: <A non-zero left module presented by 30 relations for 14 generators>
  -2: <A non-zero left module presented by 18 relations for 7 generators>
  -3: <A non-zero left module presented by 12 relations for 4 generators>
of
<A non-zero left module presented by 48 relations for 20 generators>>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
```

**Example B.3** (PurityFiltration). Example B.1 continued. This is an example for Subsections 9.1.3 and 9.1.5.

```
gap> filt := PurityFiltration( W );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0: <A codegree-[ 1, 1 ]-pure rank 2 left module presented by
         3 relations for 4 generators>
  -1: <A codegree-1-pure codim 1 left module presented by
         4 relations for 3 generators>
  -2: <A cyclic reflexively pure codim 2 left module presented by
         2 relations for a cyclic generator>
  -3: <A cyclic reflexively pure codim 3 left module presented by
```

```
         3 relations for a cyclic generator>
of
<A non-pure rank 2 left module presented by 6 relations for 5 generators>>
gap> W;
<A non-pure rank 2 left module presented by 6 relations for 5 generators>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), W );
true
gap> Source( m );
<A left module presented by 12 relations for 9 generators (locked)>
gap> Display( last );
0,  0,   x, -y,0,1, 0,    0,  0,
x*y,-y*z,-z,0, 0,0, 0,    0,  0,
x^2,-x*z,0, -z,1,0, 0,    0,  0,
0,  0,   0, 0, y,-z,0,    0,  0,
0,  0,   0, 0, x,0, -z,   0,  1,
0,  0,   0, 0, 0,x, -y,   -1, 0,
0,  0,   0, 0, 0,-y,x^2-1,0,  0,
0,  0,   0, 0, 0,0, 0,    z,  0,
0,  0,   0, 0, 0,0, 0,    y-1,0,
0,  0,   0, 0, 0,0, 0,    0,  z,
0,  0,   0, 0, 0,0, 0,    0,  y,
0,  0,   0, 0, 0,0, 0,    0,  x
Cokernel of the map

Q[x,y,z]^(1x12) --> Q[x,y,z]^(1x9),

currently represented by the above matrix
gap> Display( filt );
Degree 0:

0,  0,   x, -y,
x*y,-y*z,-z,0,
x^2,-x*z,0, -z
Cokernel of the map

Q[x,y,z]^(1x3) --> Q[x,y,z]^(1x4),

currently represented by the above matrix
----------
Degree -1:
```

```
y,-z,0,
x,0, -z,
0,x, -y,
0,-y,x^2-1
Cokernel of the map

Q[x,y,z]^(1x4) --> Q[x,y,z]^(1x3),

currently represented by the above matrix
----------
Degree -2:

Q[x,y,z]/< z, y-1 >
----------
Degree -3:

Q[x,y,z]/< z, y, x >
gap> Display( m );
1,    0,     0,  0,    0,
0,    -1,    0,  0,    0,
0,    0,     -1, 0,    0,
0,    0,     0,  -1,   0,
-x^2,-x*z,  0,  -z,   0,
0,    0,     x,  -y,   0,
0,    0,     0,  0,    -1,
0,    0,     x^2,-x*y,y,
x^3, x^2*z,0,   x*z,  -z

the map is currently represented by the above 9 x 5 matrix
```

**Example B.4** (`PurityFiltration`, *non*commutative)**.** This is a *non*commutative example for Subsections 9.1.3 and 9.1.5. Let $A_3 := \mathbb{Q}[x,y,z]\langle D_x, D_y, D_z\rangle$ be the 3-dimensional Weyl algebra.

```
gap> A3 := RingOfDerivations( Qxyz, "Dx,Dy,Dz" );;
gap> nmat := HomalgMatrix( "[ \
3*Dy*Dz-Dz^2+Dx+3*Dy-Dz,           3*Dy*Dz-Dz^2,     \
Dx*Dz+Dz^2+Dz,                     Dx*Dz+Dz^2,       \
Dx*Dy,                             0,                \
Dz^2-Dx+Dz,                        3*Dx*Dy+Dz^2,     \
Dx^2,                              0,                \
-Dz^2+Dx-Dz,                       3*Dx^2-Dz^2,      \
Dz^3-Dx*Dz+Dz^2,                   Dz^3,             \
2*x*Dz^2-2*x*Dx+2*x*Dz+3*Dx+3*Dz+3,2*x*Dz^2+3*Dx+3*Dz\
]", 8, 2, A3 );

<A homalg external 8 by 2 matrix>
```

```
gap> N := LeftPresentation( nmat );
<A left module presented by 8 relations for 2 generators>
gap> filt := PurityFiltration( N );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0: <A zero left module>
  -1: <A cyclic reflexively pure codim 1 left module presented by
         1 relation for a cyclic generator>
  -2: <A cyclic reflexively pure codim 2 left module presented by
         2 relations for a cyclic generator>
  -3: <A cyclic reflexively pure codim 3 left module presented by
         3 relations for a cyclic generator>
of
<A non-pure codim 1 left module presented by 8 relations for 2 generators>>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels [ 0 .. 2 ]
each consisting of left modules at bidegrees [ -3 .. 0 ]x[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
 . . . *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s . . .
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:
```

```
a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
 . . . *
---------
Level 1:

 * * * *
 . * * *
 . . * *
 . . . .
---------
Level 2:

 s . . .
 . s . .
 . . s .
 . . . .
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), N );
true
gap> Source( m );
<A left module presented by 6 relations for 3 generators (locked)>
gap> Display( last );
Dx,-1/3,-2/9*x,
0, Dy,  -1/3,
0, Dx,  1,
0, 0,   Dz,
0, 0,   Dy,
0, 0,   Dx
Cokernel of the map

R^(1x6) --> R^(1x3), ( for R := Q[x,y,z]<Dx,Dy,Dz> )

currently represented by the above matrix
gap> Display( filt );
Degree 0:
```

```
0
----------
Degree -1:

Q[x,y,z]<Dx,Dy,Dz>/< Dx >
----------
Degree -2:

Q[x,y,z]<Dx,Dy,Dz>/< Dy, Dx >
----------
Degree -3:

Q[x,y,z]<Dx,Dy,Dz>/< Dz, Dy, Dx >
gap> Display( m );
1,                 1,
-3*Dz-3,           -3*Dz,
-3*Dz^2+3*Dx-3*Dz,-3*Dz^2

the map is currently represented by the above 3 x 2 matrix
```

**Example B.5** (Cohomological `GrothendieckSpectralSequence`). Example B.1 continued. Compute the Tor-Ext spectral sequence for the triple $F := - \otimes W$, $G := \mathrm{Hom}(-, D)$, and the $D$-module $W$. This is an example for Subsection 9.2.1.

```
gap> F := InsertObjectInMultiFunctor( Functor_TensorProduct, 2, W, "TensorW" );
<The functor TensorW>
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable cohomological spectral sequence with sheets at levels [ 0 .. 4 ]
each consisting of left modules at bidegrees [ -3 .. 0 ]x[ 0 .. 3 ]>
gap> homalgRingStatistics(Qxyz);
rec( BasisOfRowModule := 110, BasisOfColumnModule := 16,
  BasisOfRowsCoeff := 50, BasisOfColumnsCoeff := 60, DecideZeroRows := 241,
  DecideZeroColumns := 31, DecideZeroRowsEffectively := 51,
  DecideZeroColumnsEffectively := 63, SyzygiesGeneratorsOfRows := 184,
  SyzygiesGeneratorsOfColumns := 63 )
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:
```

```
 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:
```

```
 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3: <A non-zero cyclic left module presented by
         3 relations for a cyclic generator>
  -2: <A non-zero left module presented by 17 relations for 6 generators>
  -1: <A non-zero left module presented by 19 relations for 9 generators>
   0: <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by 66 relations for 41 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3: <A non-zero cyclic left module presented by
         3 relations for a cyclic generator>
  -2: <A non-zero left module presented by 12 relations for 4 generators>
  -1: <A non-zero left module presented by 18 relations for 8 generators>
   0: <A non-zero left module presented by 11 relations for 10 generators>
of
<A left module presented by 21 relations for 12 generators>>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
```

**Example B.6** (Tor-Ext spectral sequence)**.** Here we compute the Tor-Ext spectral sequence of the bicomplex $B := \mathrm{Hom}(P^W, D) \otimes P^W$. This is an example for Subsection 9.2.2.

```
gap> P := Resolution( W );
<A right acyclic complex containing 3 morphisms of left modules at degrees
[ 0 .. 3 ]>
gap> GP := Hom( P );
<A cocomplex containing 3 morphisms of right modules at degrees [ 0 .. 3 ]>
gap> FGP := GP * P;
<A cocomplex containing 3 morphisms of left complexes at degrees [ 0 .. 3 ]>
gap> BC := HomalgBicomplex( FGP );
```

```
<A bicocomplex containing left modules at bidegrees [ 0 .. 3 ]x[ -3 .. 0 ]>
gap> p_degrees := ObjectDegreesOfBicomplex( BC )[1];
[ 0 .. 3 ]
gap> II_E := SecondSpectralSequenceWithFiltration( BC, p_degrees );
<A stable cohomological spectral sequence with sheets at levels [ 0 .. 4 ]
each consisting of left modules at bidegrees [ -3 .. 0 ]x[ 0 .. 3 ]>
gap> homalgRingStatistics(Qxyz);
rec( BasisOfRowModule := 109, BasisOfColumnModule := 1,
  BasisOfRowsCoeff := 48, BasisOfColumnsCoeff := 0, DecideZeroRows := 190,
  DecideZeroColumns := 1, DecideZeroRowsEffectively := 49,
  DecideZeroColumnsEffectively := 0, SyzygiesGeneratorsOfRows := 166,
  SyzygiesGeneratorsOfColumns := 2 )
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
```

```
Level 0:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
  -3: <A non-zero cyclic left module presented by
         3 relations for a cyclic generator>
  -2: <A non-zero left module presented by 17 relations for 7 generators>
  -1: <A non-zero left module presented by 25 relations for 12 generators>
   0: <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by 38 relations for 24 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:
```

```
  -3: <A non-zero cyclic left module presented by
3 relations for a cyclic generator>
  -2: <A non-zero left module presented by 12 relations for 4 generators>
  -1: <A non-zero left module presented by 21 relations for 8 generators>
   0: <A non-zero left module presented by 11 relations for 10 generators>
of
<A left module presented by 23 relations for 12 generators>>
gap> m := IsomorphismOfFiltration( filt );
<An isomorphism of left modules>
```

**Example B.7** (`CodegreeOfPurity`)**.** For two torsion-free $D$-modules $V$ and $W$ of rank 2 compute the three homological invariants

- projective dimension,
- AUSLANDER's degree of torsion-freeness, and
- codegree of purity

mentioned in Subsection 9.1.5 are computed. The codegree of purity is able to distinguish the two modules.

```
gap> vmat := HomalgMatrix( "[ \
0,  0,  x,-z, \
x*z,z^2,y,0,  \
x^2,x*z,0,y   \
]", 3, 4, Qxyz );
<A homalg external 3 by 4 matrix>
gap> V := LeftPresentation( vmat );
<A non-zero left module presented by 3 relations for 4 generators>
gap> wmat := HomalgMatrix( "[ \
0,  0,  x,-y, \
x*y,y*z,z,0,  \
x^2,x*z,0,z   \
]", 3, 4, Qxyz );
<A homalg external 3 by 4 matrix>
gap> W := LeftPresentation( wmat );
<A non-zero left module presented by 3 relations for 4 generators>
gap> Rank( V );
2
gap> Rank( W );
2
gap> ProjectiveDimension( V );
2
gap> ProjectiveDimension( W );
2
```

```
gap> DegreeOfTorsionFreeness( V );
1
gap> DegreeOfTorsionFreeness( W );
1
gap> CodegreeOfPurity( V );
[ 2 ]
gap> CodegreeOfPurity( W );
[ 1, 1 ]
gap> filtV := PurityFiltration( V );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:
   0: <A codegree-[ 2 ]-pure rank 2 left module presented by
         3 relations for  4 generators>
  -1: <A zero left module>
  -2: <A zero left module>
of
<A codegree-[ 2 ]-pure rank 2 left module presented by
3 relations for 4 generators>>
gap> filtW := PurityFiltration( W );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:
   0: <A codegree-[ 1, 1 ]-pure rank 2 left module presented by
         3 relations for 4 generators>
  -1: <A zero left module>
  -2: <A zero left module>
of
<A codegree-[ 1, 1 ]-pure rank 2 left module presented by
3 relations for 4 generators>>
gap> II_EV := SpectralSequence( filtV );
<A stable homological spectral sequence with sheets at levels [ 0 .. 4 ]
each consisting of left modules at bidegrees [ -3 .. 0 ]x[ 0 .. 2 ]>
gap> Display( II_EV );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 * * *
 . * *
---------
Level 1:
```

```
 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
---------
Level 1:

 * * * *
 * * * *
 . . * *
---------
Level 2:

 * . . .
 * . . .
 . . * *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
```

```
 . . . s
gap> II_EW := SpectralSequence( filtW );
<A stable homological spectral sequence with sheets at levels [ 0 .. 4 ]
each consisting of left modules at bidegrees [ -3 .. 0 ]x[ 0 .. 2 ]>
gap> Display( II_EW );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 . * *
 . . *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
```

```
 . * * *
 . . . *
---------
Level 2:

 * . . .
 . * . .
 . . . *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
 . . . s
```

An alternative title for this work could have been "Squeezing spectral sequences".