Algorithmic homological algebra

UNIVERSITÄT SIEGEN

Exercise sheet 5

Prof. Dr. Mohamed Barakat, Sebastian Posur

Exercise 1. (Computable rings)
Given a constructive field k equipped with the Gaussian normal form algorithm, more precisely, an algorithm to compute the row reduced echelon form (RREF). Apply such an algorithm to the block matrix $\left(\begin{smallmatrix} 1 & \mathtt{B} & 0 \\ 0 & \mathtt{A} & 1 \end{smallmatrix}\right)$ and obtain $\left(\begin{smallmatrix} 1 & \mathtt{B}' & -\mathtt{X} \\ 0 & \mathtt{A}' & \mathtt{Y} \\ 0 & 0 & \mathtt{S} \end{smallmatrix}\right)$. Show that setting

- $\mathtt{DecideZeroRows}(\mathtt{B}, \mathtt{A}) := \mathtt{B}'$,
- $\mathtt{DecideZeroRowsEffectively}(\mathtt{B}, \mathtt{A}) := (\mathtt{B}', -\mathtt{X})$,
- $\mathtt{SyzygiesOfRows}(\mathtt{A}) := \mathtt{S}$,

turns k into a computable ring.

Exercise 2. (Category of matrices)
Let k be a constructive field. We construct the category k-vec as follows.

1. $\mathrm{Obj}_{k\text{-vec}} := \mathbb{N}_0$.
2. For $m, n \in \mathbb{N}_0$, $\mathrm{Hom}_{k\text{-vec}}(m, n) := k^{m \times n}$.
3. For $n \in \mathbb{N}_0$, id_n is the $n \times n$ identity matrix I_n.
4. For $m, n, o \in \mathbb{N}_0$, $A \in k^{m \times n}$, $B \in k^{n \times o}$, composition is given by the matrix product $A \cdot B$.

Show that k-vec is a constructively abelian category.

Exercise 3. (Gröbner basis)

(a) Prove Gordan's lemma: Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over a field k. Then any nonempty set X of monomials in R has only finitely many minimal elements in the partial order given by divisibility.

(b) Let $f_1 = x^2y - y^3$, $f_2 = x^3 \in k[x, y]$. Compute a Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$ with respect to $>_{\mathrm{lex}}$. Visualize the monomials in $\boldsymbol{L}(I)$, and compute a multiplication table for $k[x, y]/I$.

Exercise 4. (Generalized inverses)
Prove that for any generalized morphism γ, we have $(\gamma^{-1})^{-1} = \gamma$.

Exercise 5. (Generalized morphisms and additive relations)
Given a ring R and R-modules A and B, we call a submodule $S \subseteq A \oplus B$ an **additive relation** from A to B. Given two additive relations $S \subseteq A \oplus B$ and $T \subseteq B \oplus C$, we define their composite as

$$S \cdot T := \{(a, c) \in A \oplus C \mid \exists b \in B : (a, b) \in S \wedge (b, c) \in T\}.$$

Now, let $A' \subseteq A$ and $B' \subseteq B$ be submodules. Given a generalized morphism

$$\psi : A \hookleftarrow A' \xrightarrow{\overline{\psi}} B/B' \twoheadleftarrow B,$$

where $A \hookleftarrow A'$ and $B/B' \twoheadleftarrow B$ are the canoncial inclusion and projection, we define the additive relation

$$F(\psi) := \left\{(a, b) \in A' \oplus B \mid \overline{\psi}(a) = b + B'\right\} \subseteq A \oplus B.$$

(a) Show that F defines a bijection from generalized morphisms from A to B to additive relations from A to B.

(b) Show that F is compatible with composition of generalized morphisms and composition of additive relations, i.e., given another generalized morphism ϕ from B to C, we have

$$F(\psi \cdot \phi) = F(\psi) \cdot F(\phi).$$

This exercise sheet will be discussed on 19.01.2017.