Algorithmic homological algebra

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Exercise sheet 2

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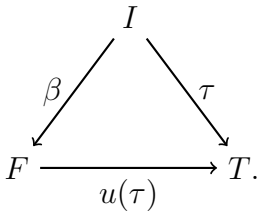
Exercise 1. (Free modules)
Let R be a ring and let I be a set. A pair $(F, \beta : I \to F)$ consisting of a left R-module F and a map $\beta : I \to F$ is called a **free module with (unordered) basis** β if it satisfies the following universal property: For every left R-module T and every map $\tau : I \to T$, there exists exactly one R-module homomorphism $u(\tau) : F \to T$ such that the following diagram commutes:

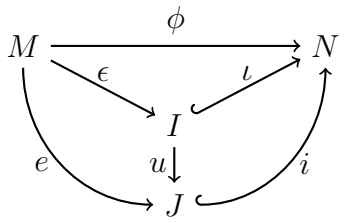


(a) Given a natural number $n \in \mathbb{N}_0$, show that for $I = \{1, \dots n\}$, the left R-module R^I (where R^I denotes the functions from I to R) can be equipped with a map $\beta : I \to R^I$ such that (R^I, β) is a free module with (unordered) basis β.

(b) Generalize (a) to the case where I is an infinite set. Hint: Consider an appropriate submodule of R^I.

(c) Given two free modules $(F, \beta : I \to F)$ and $(G, \gamma : I \to G)$, show that there exists exactly one isomorphism $\iota : F \to G$ such that $\iota \circ \beta = \gamma$.

Exercise 2. (Images)
Let $\phi : M \to N$ be a morphism. A triple (I, ϵ, ι) consisting of an object I, a morphism $\epsilon : M \to I$, and a monomorphism $\iota : I \hookrightarrow N$ such that $\iota \circ \epsilon = \phi$ is called a **mono factorization of** ϕ. A given mono factorization (I, ϵ, ι) of ϕ is called an **image of** ϕ if for every other mono factorization (J, e, i) of ϕ, there exists exactly one morphism $u : I \to J$ such that the following diagram commutes:



(a) If ϕ is an R-module homomorphism, let $\iota : \phi(M) \hookrightarrow N$ be the inclusion of the set-theoretic image of ϕ into N. Show that the pair $(\phi(M), \iota)$ can be extended to an image of ϕ.

(b) If ϕ is an R-module homomorphism, show that the object defined in the lecture

$$\mathrm{im}(\phi) := \ker(\mathrm{CokernelEpi}(\phi))$$

can be equipped with morphisms $\epsilon : M \to \mathrm{im}(\phi)$ and $\iota : \mathrm{im}(\phi) \to N$ such that $(\mathrm{im}(\phi), \epsilon, \iota)$ is an image of ϕ.

(c) Optional: Can you prove (b) in the context of an arbitrary abelian category?

Exercise 3. (Morphism from coimage to image)
Using the notation of Definition 1.2.18, show that

$$(\epsilon_\kappa \backslash \phi)/\kappa_\epsilon = \epsilon_\kappa \backslash (\phi/\kappa_\epsilon).$$

Exercise 4. (Pushout)
Construct the pushout of a given diagram $L \xleftarrow{\phi} M \xrightarrow{\psi} J$ using direct sums and cokernels.

This exercise sheet will be discussed on 10.11.2016.