Algorithmic homological algebra

Exercise sheet 1

Prof. Dr. Mohamed Barakat, Sebastian Posur

Let A, B, C be sets and let $f : A \to B$, $g : B \to C$ be functions. We introduce two conventions for the application of a function to an element and for the composition of functions:

1. **Application from the left:** We denote the image of a under f by $f(a)$. We write $g \circ f$ for the composition of f and g, i.e., $(g \circ f)(a) := g(f(a))$ for $a \in A$.
2. **Application from the right:** We denote the image of a under f by $(a)f$. We write $f \cdot g$ or simply fg for the composition of f and g, i.e., $(a)(f \cdot g) := ((a)f)g$ for $a \in A$.

Exercise 1. (Left and right modules.)
In this exercise, we introduce two definitions of a left R-module and an R-module homomorphism for a unital ring R.

1. A **left R-module** consists of the following data.

   - An abelian group $(M, +, 0)$.
   - A map $\alpha : R \times M \to M$ which we denote infix, i.e., $rm := \alpha(r, m)$ für $r \in R$ and $m \in M$.

   These data are subject to the following axioms.

   - $(rs)m = r(sm)$,
   - $1m = m$,
   - $(r + s)m = rm + sm$,
   - $r(m + n) = rm + rn$,

   for all $r, s \in R$ and $m, n \in M$. Let (M, α_M) and (N, α_N) be two left R-modules. A group homomorphism $f : M \to N$ is called an **R-module homomorphism** from (M, α_M) to (N, α_N) if $(rm)f = r(mf)$ for all $r \in R$, $m \in M$.

2. A **left R-module** consists of the following data.

   - An abelian group $(M, +, 0)$.
   - A ring homomorphism $\rho : R \to (\mathrm{End}(M), \circ)$, where $\mathrm{End}(M)$ denotes the abelian group of endomorphisms of M equipped with $\circ$ as ring multiplication.

   Let (M, ρ_M) and (N, ρ_N) be two left R-modules. A group homomorphism $f : M \to N$ is called an **R-module homomorphism** from (M, ρ_M) to (N, ρ_N) if $f \circ \rho_M = \rho_N \circ f$.

(a) Show the equivalence of both notions by mapping a left R-module (M, α_M) (corresponding to (1)) to a left R-module (M, ρ_M) (corresponding to (2)) and vice versa, such that both maps are mutal inverses that respect the notion of R-module homomorphisms (i.e., $M \to N$ is an R-module homomorphism w.r.t. (1) if and only if it is an R-module homomorphism w.r.t. (2)).

(b) Rewrite the above definitions for **right** modules and show again a correspondence analogous to (a).

(c) Show that every left R-module also is a right R^{op}-module and vice versa.

(d) Given a left $\mathbb{Z}$-module (M, ρ), we can forget its $\mathbb{Z}$-module structure and end up with an abelian group M. Show that if we are conversely given an abelian group M, we can equip it with a canonical $\mathbb{Z}$-module structure, yielding a bijective correspondence between left $\mathbb{Z}$-modules and abelian groups. Show that the same is true for right $\mathbb{Z}$-modules.

Exercise 2. (Monos and epis.)
Prove without using elements:

(a) A split epi is an epi.

(b) A split mono is a mono.

(c) The pre- and post-inverse of an isomorphism coincide, and are hence unique. We call it **the inverse**.

(d) The composition of two (split) monos is a (split) mono.

(e) The composition of two (split) epis is a (split) epi.

(f) The composition of isomorphisms is an isomorphism.

(g) If $\phi\psi$ is a mono then ϕ is a mono. The converse is false.

(h) If $\phi\psi$ is an epi then ψ is an epi. The converse is false.

Show that in the sequence of $\mathbb{Z}$-maps $\mathbb{Z} \xhookrightarrow{2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ the left morphism is mono, the right is epi, and non of them is split (as $\mathbb{Z}$-maps). If we view them as maps between sets then both become split.

This exercise sheet will be discussed on 27.10.2016.