Assignment sheet 6

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Exercise 1. (Valuation rings, 4 points)
Let R be a valuation ring. Prove that the set $\Gamma := \{I | I \trianglelefteq R\}$ of ideals in R is totally ordered w.r.t. inclusion.

Exercise 2. (Singular points, 4 points)
Let k be an algebraically closed field and let $f \in k[x_1, \dots, x_n]$ be an irreducible polynomial. A point $P = (a_1, \dots, a_n)$ of the algebraic set $V_k(f)$ is called **non-singular** iff not all formal derivatives $\partial f / \partial x_i$ vanish at P. Let $R = k[x_1, \dots, x_n]/\langle f \rangle$, and let $\mathfrak{m} = \langle \overline{x}_1 - a_1, \dots, \overline{x}_n - a_n \rangle \subset R$ be the maximal ideal corresponding to the point P. Then show:

$$P \text{ is non-singular } \iff R_\mathfrak{m} \text{ is a regular local ring.}$$

Exercise 3. (8 points)

1. Let R be a local Noetherian integral domain with maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Then the following are equivalent:
   (a) R is a DVR;
   (b) R is principal ideal domain;
   (c) $\mathfrak{m}$ is principal;
   (d) $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$;
   (e) Any ideal $\langle 0 \rangle \neq I \trianglelefteq R$ is of the form $\mathfrak{m}^n$ for some $n \in \mathbb{N}_0$;
   (f) There exists an element $p \in R$ such that any ideal $\langle 0 \rangle \neq I \trianglelefteq R$ is of the form $\langle p^n \rangle$ for some $n \in \mathbb{N}_0$;
   (g) R is normal;
   (h) $\dim_{R/\mathfrak{m}} \mathfrak{m}^k/\mathfrak{m}^{k+1} = 1$ for all $k \geq 1$.
2. Let K be a field and p a prime number. Show that $K[x]_{\langle x \rangle}$ and $\mathbb{Z}_{\langle p \rangle}$ are DVRs.

Hand in until January 23th 12:00 in the class or in Box in ENC, 2nd floor, at the entrance of the building part D.