Assignment sheet 5

Prof. Dr. Mohamed Barakat, M.Sc. Kamal Saleh

Exercise 1. (Krull dimension, 4 points)
Cosider the ring homomorphism $\varphi : k[x, y, z, u] \to k[s, t]$ defined by $x \mapsto s^3, y \mapsto s^2t, z \mapsto st^2, u \mapsto t^3$.

1. Compute $I := \ker(\varphi), \dim(I), \operatorname{codim}(I)$.

2. Give a maximal chain of prime ideals in $k[x, y, z, u]/I$.

Exercise 2. (Prime avoidance, 4 points)
Prove the following general version of prime avoidance: Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{k-2} \in \operatorname{Spec} R$ and $\mathfrak{p}_{k-1}, \mathfrak{p}_k, I \trianglelefteq R$. Then

$$I \subset \bigcup_{i=1}^{k} \mathfrak{p}_i \implies I \subset \mathfrak{p}_i \text{ for some } i.$$

Exercise 3. (Irreducible affine algebraic sets, 4 points) Let k be a field. A nonempty empty algebraic set $V = V_k(I) \subset k^n$ is called **irreducible** if it cannot be expressed as a union $V = V_1 \cup V_2$ of algebraic sets V_1, V_2 properly contained in V.

1. Show that V is irreducible if and only if $I(V)$ is prime ideal.

2. Give an example of an irreducible polynomial $f \in \mathbb{Q}[x, y]$ whose zero set $V_{\mathbb{Q}}(f)$ is not irreducible.

3. Let $V = V(y^2 - xz, x^2y - z^2, x^3 - yz) \subset \overline{\mathbb{Q}}^3$. Show that V is irreducible and compute $\dim I(V)$, $\operatorname{codim} I(V)$ and prove that $I(V)$ cannot be generated by less than 3 elements.

4. Show that $I(V)$ is isomorphic to a polynomial ring in one variable over $\overline{\mathbb{Q}}$.

Exercise 4. (Irreducible components of affine algebraic sets, 4 points)
Let X be the algebraic set in k^3 defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that X is a union of three irreducible algebraic sets. Describe them and find their prime ideals.

Hand in until January 9th 12:00 in the class or in Box in ENC, 2nd floor, at the entrance of the building part D.