Assignment sheet 4

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Exercise 1. (Noether normalization, 4 points)
Find a Noether normalization of the $\mathbb{Q}$-algebra $\mathbb{Q}[x,y,z]/\langle xy+z^2, x^2y-xy^3+z^4-1\rangle$.

Exercise 2. (Zariski theorem, 4 points)
Show the following

1. Let S/R be an integral extension, where R and S are domains. Then R is field if and only if S is a field.
2. Let $K \subset L$ be a field extension. If $L = K[\alpha_1,\dots,\alpha_n]$ is a finitely generated K-algebra, then L is finitely generated as a module over K, i.e., $[L:K] < \infty$ and L is algebraic over K.

Exercise 3. (Affine algebraic sets, 4 points)

1. Let $V = V_{\overline{\mathbb{Q}}}(I)$ be the affine algebraic set in $\overline{\mathbb{Q}}^n$ associated to some ideal I in $\overline{\mathbb{Q}}[x_1,\dots,x_n]$. Show the following statements are equivalent:
   (a) V is finite set.
   (b) The $\overline{\mathbb{Q}}$-vector space $\overline{\mathbb{Q}}[x_1,\dots,x_n]/I$ is finite-dimenstional.
2. Consider the ideals $I_1 = \langle x^2+y^2\rangle, I_2 = \langle x^2+y^2, x-y\rangle, I_3 = \langle x^2+y^2, x-y, 1-2xy\rangle$ in $\overline{\mathbb{Q}}[x_1,\dots,x_n]$. Which $V_{\overline{\mathbb{Q}}}(I_i), i \in \{1,2,3\}$ is a finite set? Why?

Exercise 4. (Hilbert Nullstellensatz, 4 points)

1. Let k be a field for which the weak (or strong) Nullstellensatz is valid. Prove that k is algebraically closed.
2. Prove that the strong Nullstellensatz implies the weak Nullstellensatz.

Hand in until December 19th 12:00 in the class or in Box in ENC, 2nd floor, at the entrance of the building part D.