Assignment sheet 3

Prof. Dr. Mohamed Barakat, M.Sc. Kamal Saleh

Exercise 1. (Integral extensions, 3 points)
Let $R \subset S$ be a ring extension and $U \subset R$ a multiplicatively closed set. Show the following

1. Show without using Proposition 5.22. that $S[U^{-1}]$ is integral over $R[U^{-1}]$.
2. If $S \setminus R$ is closed under multiplication, then R is integrally closed in S.
3. If $r \in R$ is a unit in S then it is a unit in R.

Exercise 2. (Zariski topology, 2 points)
Let $R \subset S$ be an integral ring extension. Show that the map

$$\mathrm{Spec}(S) \to \mathrm{Spec}(R), \mathfrak{P} \mapsto \mathfrak{P}^c = \mathfrak{P} \cap R$$

is closed, i.e., it maps closed sets to closed sets (in the Zariski topologies on Spec(S) and Spec(R)).

Exercise 3. (Going-up, Going-down property, 4 points)

1. Check the Going-up property for the following ring extension

$$\phi : k[s] \to k[s,t]/\langle s^2t - s, st^2 - t\rangle, s \mapsto \overline{s}.$$

2. Check the Going-down property for the following ring extension

$$\psi : k[s] \to k[s,t]/\langle st\rangle, s \mapsto \overline{s}.$$

and interpret your answers geometrically.

Exercise 4. (Going-down property fails, 4 points)
Consider the ring homomorphism $\phi : \mathbb{Q}[x,y,z] \to \mathbb{Q}[s,t]$ with $x \mapsto s, y \mapsto t^2 - s - 1, z \mapsto t^3 - t + s$. Show that the going-down property fails for the ring extension $\iota : \mathbb{Q}[x,y,z]/\ker\phi \to \mathbb{Q}[s,t]$.
Hint: Use the following two prime ideals $\mathfrak{p}_1 = \langle \overline{x}^2 - \overline{x} - \overline{y} - 1, \overline{x}^3 - \overline{z}\rangle, \mathfrak{p}_2 = \langle \overline{x} - 1, \overline{y} + 1, \overline{z} - 1\rangle$ in $\mathbb{Q}[x,y,z]/\ker\phi$. Then use Computeralgebra.

Demonstration of Example 5.21 in `homalg`[1]: Let I be an ideal of R, and let $\overline{f}_1 = f_1 + I, \dots, \overline{f}_m = f_m + I \in S := R/I$. Consider the homomorphism

$$\phi : k[y_1, \dots, y_m] \to S = R/I,\ y_i \mapsto \overline{f}_i.$$

[1] You need the latest versions of the `homalg` project packages which you can download from `https://github.com:homalg-project/homalg_project`

If J is the ideal

$$J = I\,k[\boldsymbol{x},\boldsymbol{y}] + \langle f_1 - y_1, \ldots, f_m - y_m \rangle \subset k[\boldsymbol{x},\boldsymbol{y}]$$

then by Proposition 1.55,

$$\ker\phi = \phi^{-1}(\{0\}) = J \cap k[\boldsymbol{y}].$$

We know that the ideals of S are the epimorphic images of the ideals $I' \trianglelefteq R$ under the natural epimorphism $\pi : R \to R/I = S$. Denote by $\overline{I'}$ the image of I' under π.
For an ideal $\overline{I'} \trianglelefteq S$, we define the ideal

$$J' = I'\,k[\boldsymbol{x},\boldsymbol{y}] + \langle f_1 - y_1, \ldots, f_m - y_m \rangle \subset k[\boldsymbol{x},\boldsymbol{y}].$$

Then one can show analogously to Proposition 1.55 that

$$\phi^{-1}(\overline{I'}) = J' \cap k[\boldsymbol{y}].$$

This mechanism can be used to compute contractions of the ideals of S in $k[\boldsymbol{y}]$ or $k[\boldsymbol{y}]/\ker\phi$. Now consider the ring extension $\psi : k[\boldsymbol{y}]/\ker\phi \hookrightarrow S$. For any prime ideal $\mathfrak{p} \lhd k[\boldsymbol{y}]/\ker\phi$, we can compute the prime ideals $\mathfrak{P} \lhd S$ that lie over $\mathfrak{p}$ by computing the associated primes of the extension ideal of $\mathfrak{p}$, i.e., of the ideal $\mathfrak{p}^e := \langle\psi(\mathfrak{p})\rangle \lhd S$ (See Theorem 5.9). Now let us apply this to Example 5.21 in the lecture notes.

Gap
```
gap> LoadPackage( "RingsForHomalg" );
gap> LoadPackage( "Modules" );
gap> Q := HomalgFieldOfRationalsInSingular( );
Q
gap> Q_xyz := Q * "x,y,z";
gap> Q_xyz_st := Q_xyz * "s,t";
gap> DisplayRing( Q_xyz_st );
polynomial ring, over a field, global ordering
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    x y z s t
//        block   2 : ordering C
```

Now we define another ring that has the required block ordering $\{s > t\} > \{x > y > z\}$.

Gap
```
gap> Q_xyz_st_po := PolynomialRingWithProductOrdering( Q_xyz_st );
gap> DisplayRing( Q_xyz_st_po );
polynomial ring, over a field, global ordering
//   characteristic : 0
//   number of vars : 5
//        block   1 : ordering dp
//                  : names    s t
//        block   2 : ordering dp
//                  : names    x y z
//        block   3 : ordering C
```

Let us compute first the kernel of the given k-algebra homomorphism

$$\varphi : k[x, y, z] \to k[s, t],\ x \mapsto s, y \mapsto t^2 - 1, z \mapsto t(t^2 - 1).$$

Gap
```
gap> J := LeftSubmodule( "x-s,y-(t^2-1),z-t*(t^2-1)", Q_xyz_st_po );
gap> OnBasisOfPresentation( J );
gap> Display( J );
y^3+y^2-z^2,
t*z-y^2-y,
t*y-z,
s-x,
t^2-y-1

A (left) ideal generated by the 5 entries of the above matrix
```

This means that $\ker \varphi = \langle y^3 + y^2 - z^2 \rangle \lhd k[x, y, z]$ and that the induced ring extension

$$\iota : R = k[x, y, z]/\langle z^2 - y^2(y + 1)\rangle \hookrightarrow k[s, t] = S$$

is integral (See Exercise 4 in the second assignment sheet).
Now we want to compute the contraction of the ideal $\mathfrak{P}_1 = \langle s - t\rangle \lhd k[s, t]$:

Gap
```
gap> J1 := LeftSubmodule( "s-t,x-s,y-(t^2-1),z-t*(t^2-1)", Q_xyz_st_po );
gap> OnBasisOfPresentation( J1 );
gap> Display( J );
y^2-x*z+y,
x*y-z,
x^2-y-1,
t-x,
s-x

A (left) ideal generated by the 5 entries of the above matrix
```

Hence, the contraction of $\mathfrak{P}_1$ is

$$\mathfrak{p}_1 := \mathfrak{P}_1^c = \langle \overline{y}^2 - \overline{x}\,\overline{z} + \overline{y}, \overline{x}\,\overline{y} - \overline{z}, \overline{x}^2 - \overline{y} - 1\rangle \lhd R.$$

It has been said in the lecture notes that

$$\mathfrak{P}_1^c = \langle \overline{x}^2 - 1 - \overline{y}, \overline{x}(\overline{x}^2 - 1) - \overline{z}\rangle \lhd R.$$

So let us make sure that the two ideals are actually equal:

Gap
```
gap> ker_phi := LeftSubmodule( "y3+y2-z2", Q_xyz );
gap> R := Q_xyz / ker_phi;
gap> p1 := LeftSubmodule( "y^2-x*z+y,x*y-z,x^2-y-1", R );
gap> q1 := LeftSubmodule( "x2-1-y,x*(x2-1)-z", R );
gap> p1=q1;
true
```

Then we defined the maximal ideal $\mathfrak{p}_2 := \langle \overline{x}-1, \overline{y}, \overline{z} \rangle \lhd R$ and said that $\mathfrak{p}_1 \subset \mathfrak{p}_2$. Let us verify this:

Gap
```
gap> p2 := LeftSubmodule( "x-1,y,z", R );
gap> IsSubset( p2, p1 );
true
```

We want to compute all prime ideals that lie over $\mathfrak{p}_2$. To do this, we need to compute the associated primes of the extension $\mathfrak{p}_2^e = \langle \iota(\mathfrak{p}_2) \rangle$ in S.

Gap
```
gap> S := Q * "s,t";
Q[s,t]
gap> iota := RingMap( HomalgMatrix( "s,t^2-1,t*(t^2-1)",3,1,S), Q_xyz, S );
<A "homomorphism" of rings>
gap> p2_e_mat := Pullback( iota, HomalgMatrix( "x-1,y,z", 3, 1, Q_xyz ) );
<A 3 x 1 matrix over an external ring>
gap> Display( p2_e_mat );
s-1,
t^2-1,
t^3-t
gap> iota_p2 := LeftSubmodule( p2_e_mat );
gap> ass_ideals := RadicalDecomposition( iota_p2 );
gap> Display( ass_ideals[ 1 ] );
t-1,
s-1

A (left) ideal generated by the 2 entries of the above matrix
gap> Display( ass_ideals[ 2 ] );
t+1,
s-1

A (left) ideal generated by the 2 entries of the above matrix
```

Hence there exists exactly two maximal ideals of S lying over $\mathfrak{p}_2$, namely $\langle s-1, t-1 \rangle$ and $\langle s-1, t+1 \rangle$. However, $\mathfrak{P}_2 := \langle s-1, t+1 \rangle$ does not contain $\mathfrak{P}_1 := \langle s-t \rangle$.

Gap
```
gap>  P1 := LeftSubmodule( "s-t", S );
<A principal torsion-free (left) ideal given by a cyclic generator>
gap>  P2 := LeftSubmodule( "s-1,t+1", S );
<A torsion-free (left) ideal given by 2 generators>
gap>  IsSubset( P2, P1 );
false
```

Hand in until December 5th 12:00 in the class or in Box in ENC, 2nd floor, at the entrance of the building part D.