# Assignment sheet 2

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**Exercise 1. (Primary decomposition, 4 points)**
Let $R = \mathbb{C}[x, y, z]$, $I = \langle xy^2, x^2 + y^2 - z^2, z - 3\rangle \trianglelefteq R$ and $f = y \in R$. Extensions and contractions are understood w.r.t. the localization morphism $R \to R_f$.

1. Compute using some CAS[1] a minimal primary decomposition of $I$.
2. Using the previous primary decomposition,
   (a) compute $\mathrm{Ass}(I), \min \mathrm{Ass}(I)$ and determine which associated primes of $I$ are embedded and which are isolated.
   (b) compute a minimal primary decomposition and a generating system of both ideals $I^e \trianglelefteq R_f$ and $I^{ec} \trianglelefteq R$.
   (c) compute a generating system for $\sqrt{I}$.
   (d) give an example of a proper ideal $J \triangleleft R$ with $J^{ec} = J$.

**Exercise 2. (Kronecker theorem, 4 points)**

1. Let $S/R$ be a ring extension and $I \trianglelefteq R$. Prove that for $s \in S$ the following are equivalent:
   (a) $s$ is integral over $I$;
   (b) $R[s]$ is finite over $R$ and $s \in \sqrt{I \cdot R[s]}$;
   (c) $R[s]$ is contained in a commutative subring $S' \leq S$ which is finite over $R$ and $s \in \sqrt{I \cdot S'}$;
2. Let $k$ be a field. Why are both ring extensions

$$R := k[y] \leq k[x, y]/\langle xy - 1\rangle =: S$$

   and

$$R := k[y] \leq k[x, y]/\langle xy\rangle =: S'$$

   not integral?

**Exercise 3. (Gaussian integers rings, 4 points)** Consider the ring extension $R := \mathbb{Z} \subset \mathbb{Z}[\sqrt{-5}] =: S$.

1. Show that $S$ is finite over $R$.

[1] You can use the command `PrimaryDecomposition` in the `homalg` project.

2. Find all ideals in $S$ lying over $\mathfrak{p} = \langle i \rangle \lhd R$ for $i \in \{3, 5, 11\}$.

**Exercise 4. (Integral ring extensions, 4 points)** Let $k$ be a field. Prove that:

1. A UFD is normal.
2. $k[x, y, z]/\langle x^2 - y^2 z \rangle$ is not normal.
3. Let $\iota : k[x, y, z]/\langle x^2 - y^2 z \rangle \to k[s, t]$ be a ring homomorphism defined by $\overline{x} \mapsto st, \overline{y} \mapsto t, \overline{z} \mapsto s^2$. Then $\iota$ is injective and it defines an integral ring extension.
4. $k[x, y]/\langle xy \rangle$ ist not integral over $k[x]$ but over $k[x + y]$.
5. Let $R = k[x]$ and $S = R[y]/I$ with $I = \langle xy - 1 \rangle \cap \langle x, y \rangle \subseteq R[y]$. Show that $R \subset S$ is not an integral ring extension.

*You can use without proof*: With notations as in Proposition 1.55 in the lecture notes, we can view $k[\boldsymbol{y}]/\ker \varphi$ as a subring of $S$. This ring extension is integral iff for each $1 \leq i \leq n$ there is an element of the Gröbner basis of $J$ whose leading monomial is of type $x_i^{\alpha_i}$ for some $\alpha_i \geq 1$.

Hand in until November 21th 12:00 in the class or in Box in ENC, 2nd floor, at the entrance of the building part D.