Assignment sheet 1

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Exercise 1. (Zariski topology, 6 points)
Let R be a ring. For every ideal $I \trianglelefteq R$ we define the **zero locus** of I to be the set

$$\mathcal{V}(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq I\} \subseteq \operatorname{Spec} R.$$

Show the following:

1. If $\{I_i\}$ is a family of ideals of R then $\bigcap_i \mathcal{V}(I_i) = \mathcal{V}(\sum_i I_i)$.
2. If $I_1, I_2 \lhd R$ then
   a. $\mathcal{V}(I_1 \cap I_2) = \mathcal{V}(I_1 I_2) = \mathcal{V}(I_1) \cup \mathcal{V}(I_2)$.
   b. $I_2 \subseteq I_1$ implies $\mathcal{V}(I_1) \subseteq \mathcal{V}(I_2)$.
   c. $\mathcal{V}(I_1) \subseteq \mathcal{V}(I_2)$ iff $\sqrt{I_2} \subseteq \sqrt{I_1}$.
3. Show that we can define a topology on $\operatorname{Spec} R$ by taking the subsets of the form $\mathcal{V}(I)$ for $I \lhd R$ as the closed subsets. We call this topology the **Zariski topology** on $\operatorname{Spec} R$.
4. For $f \in R$, let
   $$D(f) := D_R(f) := \operatorname{Spec} R \setminus \mathcal{V}(\langle f \rangle)$$
   be the open set of prime ideals not containing f. Open sets of this form are called **distinguished open sets of** $\operatorname{Spec} R$. Show they form a basis of the Zariski topology of $\operatorname{Spec} R$.
5. Let $f \in R$. Show that $D(f) = \emptyset$ iff f is nilpotent.

Exercise 2. (Closed sets in Spec of principal ideal domains, 4 points)
Let R be a principal ideal domain. Show the following

1. All prime ideals in R are maximal or the zero ideal.
2. There is $1-1$ correspondence between the closed points of $\operatorname{Spec} R$ and the equivalence classes of prime elements $p \in R$, where $p \sim p'$ if there is a unit $u \in R^\times$ with $p' = up$.
3. The closed sets $\neq \operatorname{Spec} R$ are the finite sets consisting of closed points.
4. Give a precise description for the closed sets in $\operatorname{Spec} R$ for $R = 0, \mathbb{Z}, \mathbb{C}, \mathbb{C}[x], \mathbb{Q}[x]$.

Exercise 3. (4 points)
Let R be a ring. For every subset $Y \subseteq \operatorname{Spec} R$ we set

$$\mathcal{I}(Y) := \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}.$$

Show the following:

1. $\sqrt{\mathcal{I}(Y)} = \mathcal{I}(Y)$.
2. $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$ for any $I \lhd R$.
3. $\mathcal{V}(\mathcal{I}(Y)) = \overline{Y}$, where $\overline{Y}$ denotes the closure of Y in $\operatorname{Spec} R$.
4. There is $1-1$ correspondence between the set of radical ideals in R and the closed sets of $\operatorname{Spec} R$.

Hand in until November 7th 12:00 in the class or in Box in ENC, 2nd floor, at the entrance of the building part D.