Assignment sheet 11

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Exercise 1. (Tensor product of finitely presented modules)
Let R be a commutative unitary computable ring. Let A be an $m \times n$ matrix and B a $p \times q$ matrix with entries in R. The **Kronecker product** $A \otimes B$ is the $mp \times nq$ block matrix:

$$A \otimes B := \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}.$$

For any two finitely generated R-modules $M = \langle m_1, \dots, m_r \rangle_R, N = \langle n_1, \dots, n_s \rangle_R$, the set $\{m_1 \otimes n_1, m_1 \otimes n_2, \dots, m_r \otimes n_s\}$ generates $M \otimes_R N$. We call this set the **product generating set** of $M \otimes_R N$.

1. Let $f : R^m \xrightarrow{A} R^n$, $g : R^p \xrightarrow{B} R^q$ be two R-linear maps defined by the matrices A and B, respectively. Show that the matrix of $f \otimes g : R^m \otimes_R R^p \to R^n \otimes_R R^q$ w.r.t. the product bases of $R^m \otimes_R R^p$ and $R^n \otimes_R R^q$ is $A \otimes B$.

2. Let $N \in R$-fpmod with presentation matrix $\mathtt{N} \in R^{u \times s}$ and R^p be the free R-module of rank p. Use 1. and the fact that $R^p \otimes -$ is right exact to compute a presentation matrix for $R^p \otimes_R N$.

3. Let $h : R^p \xrightarrow{H} R^q$ be an R-linear map in R-fpmod. Use 2. to compute a morphism in R-fpres corresponding to $h \otimes \mathrm{id}_N : R^p \otimes_R N \to R^q \otimes_R N$.

4. Let $M \in R$-fpmod with presentation matrix $\mathtt{M} \in R^{p \times q}$. Use 3. and the fact that $- \otimes N$ is right exact to compute a presentation matrix for $M \otimes_R N$.

5. (a) Let $R = \mathbb{Z}$. Apply the above to compute $\mathbb{Z}/4\mathbb{Z} \otimes \mathbb{Z}/6\mathbb{Z}$.

   (b) Let $R = \mathbb{Q}[x, y, z]$, $I := \langle x^2 - y^2, xz - y, -x^5 + yz \rangle_R$, $M := \operatorname{coker}(\left(\begin{smallmatrix} x & y \\ z & y \end{smallmatrix}\right))$. Compute $\operatorname{Ann}_R(R/I \otimes_R M)$.

Exercise 2. (Being a flat R-module is local property)
Let R be a ring, M an R-module. The following are equivalent:

1. M is a flat R-module.

2. $M_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec} R$.

3. $M_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module for all $\mathfrak{m} \in \operatorname{Max} R$.

Hand in until Juli 25th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.