Assignment sheet 10

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Exercise 1. (5-Lemma and split exact sequences)

1. Prove the 5-Lemma: Let R be a ring and

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{\varphi_1} & M_2 & \xrightarrow{\varphi_2} & M_3 & \xrightarrow{\varphi_3} & M_4 & \xrightarrow{\varphi_4} & M_5 \\
\alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\
N_1 & \xrightarrow[\psi_1]{} & N_2 & \xrightarrow[\psi_2]{} & N_3 & \xrightarrow[\psi_3]{} & N_4 & \xrightarrow[\psi_4]{} & N_5
\end{array}$$

   be a commutative diagram of R-modules with exact rows. Suppose α_2 and α_4 are both isomorphisms, that α_1 is an epimorphism and that α_5 is a monomorphism. Show that α_3 is isomorphism.

2. Use the 5-lemma to show that for any short exact sequences $0 \to N' \xrightarrow{\varphi} N \xrightarrow{\psi} N'' \to 0$ in the category of R-modules, the following are equivalent:

   (a) φ is a split mono.

   (b) ψ is a split epi.

   If these conditions are fulfilled, then $N \cong N' \oplus N''$ and we call the exact sequence **split**.

Remark. Any small abelian category has an exact embedding into the category of abelian groups. Hence, the previous statements are true in any small abelian category.

Exercise 2. (Hom-tensor adjunction)
Let M, N, L be R-modules. Then the R-linear map

$$\mathrm{Hom}_R(M \otimes N, L) \xrightarrow{\cong} \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$$
$$\varphi \mapsto \varphi^{\mathrm{c}} : \begin{cases} M \to \mathrm{Hom}_R(N, L) \\ m \mapsto (N \to L, n \mapsto \varphi(m \otimes n)) \end{cases}$$

is an isomorphism.

Exercise 3. (Hom-functor is left exact)
Let $\mathcal{A}$ be an abelian category and $N \in \mathcal{A}$. Show that

1. The Hom-functor $\mathrm{Hom}_{\mathcal{A}}(-, N) : \mathcal{A}^{\mathrm{op}} \to (\mathrm{Ab})$ is left exact.
2. The Hom-functor $\mathrm{Hom}_{\mathcal{A}}(N, -) : \mathcal{A} \to (\mathrm{Ab})$ is left exact.
3. The Hom-functor is generally not exact in any of its arguments.
4. Prove the converse of Proposition 2.87.1 for the category $\mathcal{A} := R$-Mod of R-modules, i.e., the sequence $M' \to M \to M'' \to 0$ is exact iff the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M'', N) \to \mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{A}}(M', N)$$

is exact for all $N \in \mathcal{A}$.

Exercise 4. (Tensor product of modules)
Let R be a ring and N an R-module. Show the following

1. Free R-modules of finite rank are flat.
2. $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/\gcd(m, n)\mathbb{Z}$.
3. The tensor functor is generally not exact in any of its arguments.

Hand in until Juli 18th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.