Assignment sheet 9

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Exercise 1. (Computing intersection and sum categorically, 6 points)
Let R be a computable commutative ring. In the lecture we showed that the essentially surjective functor

$$\mathbf{coker} : \begin{cases} R\text{-fpres} & \to R\text{-fpmod}, \\ \mathtt{M} & \mapsto M = \operatorname{coker} \mathtt{M}, \\ \kappa = (\mathtt{M}, \mathtt{A}, \mathtt{N}) & \mapsto \varphi_{\mathtt{A}} : M \to N = \operatorname{coker} \mathtt{N} \end{cases} \tag{*}$$

is fully faithful and is hence an equivalence of categories $R\text{-fpres} \cong R\text{-fpmod}$. Define $\mathtt{N}$ to be the 0×1 matrix in R-fpres and identify $\mathbf{coker}(\mathtt{N}) = R$.

1. Let $I = \langle f_1, f_2, \ldots, f_m \rangle_R$ be an ideal in R. Construct a mono $\kappa_{\mathtt{I}} = (\mathtt{M}_{\mathtt{I}}, \mathtt{A}_{\mathtt{I}}, \mathtt{N}) \in R$-fpres which maps to the subobject $I \hookrightarrow R$ under the above equivalence of categories. Prove:

   - For any such mono the entries of the column matrix $\mathtt{A}_{\mathtt{I}}$ generate the ideal I.
   - $\mathtt{A}_{\mathtt{I}}$ can be set to $\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$.

   Hint: $\kappa_{\mathtt{I}}$ can be computed as an image embedding.

2. Let $J = \langle g_1, g_2, \ldots, g_n \rangle_R$ be another ideal in R and $\kappa_{\mathtt{J}} = (\mathtt{M}_{\mathtt{J}}, \mathtt{A}_{\mathtt{J}}, \mathtt{N})$ as in 1. Moreover, let

$$\begin{array}{ccc} \mathtt{M}_{\mathtt{I}} & \xrightarrow{\mathtt{A}_{\mathtt{I}}} & \mathtt{N} \\ {\scriptstyle \mathtt{B}_{\mathtt{I}}}\uparrow & & \uparrow{\scriptstyle \mathtt{A}_{\mathtt{J}}} \\ \mathtt{P}_{\mathtt{I},\mathtt{J}} & \xrightarrow[\mathtt{B}_{\mathtt{J}}]{} & \mathtt{M}_{\mathtt{J}} \end{array}$$

   be the pullback diagram of $\kappa_{\mathtt{I}}, \kappa_{\mathtt{J}}$, i.e., $\mathtt{P}_{\mathtt{I},\mathtt{J}}$ is the categorical intersection of the subobjects $\kappa_{\mathtt{I}}, \kappa_{\mathtt{J}}$. We know from the previous exercise sheet that $(\mathtt{P}_{\mathtt{I},\mathtt{J}}, \mathtt{B}_{\mathtt{I}}\mathtt{A}_{\mathtt{I}}, \mathtt{N})$ is the mono corresponding to the embedding $I \cap J \hookrightarrow R$ under the above equivalence (*). Now let

$$\begin{array}{ccc} \mathtt{M_I} & \xrightarrow{\mathtt{C_I}} & \mathtt{Q_{I,J}} \\ {\scriptstyle \mathtt{B_I}}\uparrow & & \uparrow{\scriptstyle \mathtt{C_J}} \\ \mathtt{P_{I,J}} & \xrightarrow[\mathtt{B_J}]{} & \mathtt{M_J} \end{array}$$

be the pushout diagram of $(\mathtt{P_{I,J}}, \mathtt{B_I}, \mathtt{M_I}), (\mathtt{P_{I,J}}, \mathtt{B_J}, \mathtt{M_J})$. We know that $(\mathtt{P_{I,J}}, \mathtt{B_I A_I}, \mathtt{N}) \sim (\mathtt{P_{I,J}}, \mathtt{B_J A_J}, \mathtt{N})$, hence there is a unique morphism $\mathtt{d} := (\mathtt{Q_{I,J}}, \mathtt{D}, \mathtt{N})$ such that $(\mathtt{M_I}, \mathtt{C_I D}, \mathtt{N}) \sim (\mathtt{M_I}, \mathtt{A_I}, \mathtt{N})$ and $(\mathtt{M_J}, \mathtt{C_J D}, \mathtt{N}) \sim (\mathtt{M_J}, \mathtt{A_J}, \mathtt{N})$. Show that $\mathtt{d}$ is a mono corresponding to the embedding $I + J \hookrightarrow R$ under the above equivalence of categories.

3. Compute using the above categorical algorithms $I \cap J$ and $I + J$ for
   (a) $R := \mathbb{Z}, I := \langle 4 \rangle_{\mathbb{Z}} = 4\mathbb{Z}$, and $J := \langle 6 \rangle_{\mathbb{Z}} = 6\mathbb{Z}$.
   (b) $R := \mathbb{Q}[x, y, z], I := \langle x^2 - y^2, xz - y, -x^5 + yz \rangle$, and $J := \langle -x^2y + y^3 + xz^2 - yz, y^3 - xyz + xz^2 - yz \rangle$.
4. Construct a presentation matrix for the R-module $(I + J)/I$.
5. Generalize the above to submodules of finitely presented modules over a left computable ring R.

Exercise 2. (Computing annihilators categorically, 4 points)

1. Let R be a left computable ring, $\mathtt{N} \in R$-fpres the zero matrix of dimensions 0×1, and $\mathtt{M} \in R$-fpres of dimensions $m \times n$. For every $1 \leq i \leq n$, denote by $\mathtt{E_i}$ the i-th standard basis row vector with n columns, and by $\epsilon_{\mathtt{i}}$ the morphism $(\mathtt{N}, \mathtt{E_i}, \mathtt{M})$. Suppose $\mathtt{S} := \bigcap_{i=1}^{n} \ker(\epsilon_{\mathtt{i}})$ and $\kappa := (\mathtt{S}, \mathtt{A}, \mathtt{N})$ is the embedding of $\mathtt{S}$ as a subobject of $\mathtt{N}$. Prove that κ corresponds under the equivalence (*) to the embedding $\mathrm{Ann}_R(M) \hookrightarrow R$. In particular, the entries of the column matrix $\mathtt{A}$ generate the annihilator of $M := \mathrm{coker}(\mathtt{M})$.
2. Let $R := \mathbb{Q}[x, y, z]$. Compute a generating set for the annihilator of $M := \mathrm{coker}(\mathtt{M})$, where

$$\mathtt{M} := \begin{pmatrix} x & y \\ z & y \\ x^2 - y^2 & 0 \\ 0 & x^2 - y^2 \\ xz - y & 0 \\ 0 & xz - y \\ -x^5 + yz & 0 \\ 0 & -x^5 + yz \end{pmatrix}.$$

Exercise 3. (Computing ideal quotient categorically, 6 points)
Let R be a computable ring and I, J be two finitely generated ideals in R. Show the following

1. $I : J = \mathrm{Ann}_R((I + J)/I)$.
2. $J \subset I$ then $I : J = R$.
3. $I : (J + K) = (I : J) \cap (I : K)$.
4. If R is integral domain, then $I : (r) = \frac{1}{r}(I \cap (r))$.
5. Use 3. and 4. to derived a Gröbner-basis based algorithm to compute $I : J$ when R is a polynomial ring over a field.
6. Give a categorical method to compute $I : J$ over any computable commutative unitial ring.
7. Compute $I : J$ for the ideals in Exercise 1.

Remark. For 5., use elimination theory. For 6., use Exercises 1 and 2.

Hand in until Juli 11th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.