

Assignment sheet 9

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Exercise 1. (Computing intersection and sum categorically, 6 points)

Let R be a computable commutative ring. In the lecture we showed that the essentially surjective functor

$$\mathbf{coker} : \begin{cases} R\text{-fpres} & \rightarrow R\text{-fpm}od, \\ M & \mapsto M = \mathbf{coker} M, \\ \kappa = (M, A, N) & \mapsto \varphi_A : M \rightarrow N = \mathbf{coker} N \end{cases} \quad (*)$$

is fully faithful and is hence an equivalence of categories $R\text{-fpres} \cong R\text{-fpm}od$. Define \mathbf{N} to be the 0×1 matrix in $R\text{-fpres}$ and identify $\mathbf{coker}(\mathbf{N}) = R$.

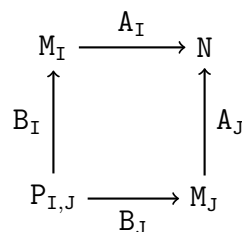
- Let $I = \langle f_1, f_2, \dots, f_m \rangle_R$ be an ideal in R . Construct a mono $\kappa_I = (M_I, A_I, N) \in R\text{-fpres}$ which maps to the subobject $I \hookrightarrow R$ under the above equivalence of categories. Prove:

- For any such mono the entries of the column matrix A_I generate the ideal I .

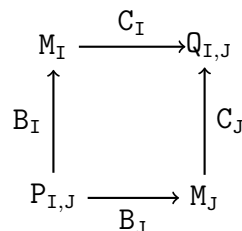
- A_I can be set to $\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$.

Hint: κ_I can be computed as an image embedding.

- Let $J = \langle g_1, g_2, \dots, g_n \rangle_R$ be another ideal in R and $\kappa_J = (M_J, A_J, N)$ as in 1. Moreover, let



be the pullback diagram of κ_I, κ_J , i.e., $P_{I,J}$ is the categorical intersection of the subobjects κ_I, κ_J . We know from the previous exercise sheet that $(P_{I,J}, B_I A_I, N)$ is the mono corresponding to the embedding $I \cap J \hookrightarrow R$ under the above equivalence (*). Now let



be the pushout diagram of $(P_{I,J}, B_I, M_I), (P_{I,J}, B_J, M_J)$. We know that $(P_{I,J}, B_I A_I, N) \sim (P_{I,J}, B_J A_J, N)$, hence there is a unique morphism $d := (Q_{I,J}, D, N)$ such that $(M_I, C_I D, N) \sim (M_I, A_I, N)$ and $(M_J, C_J D, N) \sim (M_J, A_J, N)$. Show that d is a mono corresponding to the embedding $I + J \hookrightarrow R$ under the above equivalence of categories.

3. Compute using the above categorical algorithms $I \cap J$ and $I + J$ for
 - (a) $R := \mathbb{Z}, I := \langle 4 \rangle_{\mathbb{Z}} = 4\mathbb{Z}$, and $J := \langle 6 \rangle_{\mathbb{Z}} = 6\mathbb{Z}$.
 - (b) $R := \mathbb{Q}[x, y, z], I := \langle x^2 - y^2, xz - y, -x^5 + yz \rangle$, and $J := \langle -x^2 y + y^3 + xz^2 - yz, y^3 - xyz + xz^2 - yz \rangle$.
4. Construct a presentation matrix for the R -module $(I + J)/I$.
5. Generalize the above to submodules of finitely presented modules over a left computable ring R .

Exercise 2. (Computing annihilators categorically, 4 points)

1. Let R be a left computable ring, $N \in R\text{-fpres}$ the zero matrix of dimensions 0×1 , and $M \in R\text{-fpres}$ of dimensions $m \times n$. For every $1 \leq i \leq n$, denote by E_i the i -th standard basis row vector with n columns, and by ϵ_i the morphism (N, E_i, M) . Suppose $S := \bigcap_{i=1}^n \ker(\epsilon_i)$ and $\kappa := (S, A, N)$ is the embedding of S as a subobject of N . Prove that κ corresponds under the equivalence $(*)$ to the embedding $\text{Ann}_R(M) \hookrightarrow R$. In particular, the entries of the column matrix A generate the annihilator of $M := \text{coker}(M)$.
2. Let $R := \mathbb{Q}[x, y, z]$. Compute a generating set for the annihilator of $M := \text{coker}(M)$, where

$$M := \begin{pmatrix} x & y \\ z & y \\ x^2 - y^2 & 0 \\ 0 & x^2 - y^2 \\ xz - y & 0 \\ 0 & xz - y \\ -x^5 + yz & 0 \\ 0 & -x^5 + yz \end{pmatrix}.$$

Exercise 3. (Computing ideal quotient categorically, 6 points)

Let R be a computable ring and I, J be two finitely generated ideals in R . Show the following

1. $I : J = \text{Ann}_R((I + J)/I)$.
2. $J \subset I$ then $I : J = R$.
3. $I : (J + K) = (I : J) \cap (I : K)$.
4. If R is integral domain, then $I : (r) = \frac{1}{r}(I \cap (r))$.
5. Use 3. and 4. to derived a Gröbner-basis based algorithm to compute $I : J$ when R is a polynomial ring over a field.
6. Give a categorical method to compute $I : J$ over any computable commutative unital ring.
7. Compute $I : J$ for the ideals in Exercise 1.

Remark. For 5., use elimination theory. For 6., use Exercises 1 and 2.

Hand in until Juli 11th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.