Assignment sheet 8

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Exercise 1. (Working with `homalg`)
The `homalg` project is a multi-author multi-package open source software project for constructive homological algebra. Mainly written in GAP4 it allows the use of external programs and other computer algebra systems (CASs) for specific time critical tasks. Although the central part of the source code is the formalization of abstract notions like Abelian categories, The focus lies on concrete applications ranging from linear control theory to commutative algebra and algebraic geometry.
The `homalg` project provides the possibility to compute in a lot of rings, including the ring of integers $\mathbb{Z}$, its residue class rings $\mathbb{Z}/m\mathbb{Z}$, the field of rationals $\mathbb{Q}$, polynomial rings $R[x_1, \ldots, x_n]$, field of fractions of polynomial rings $K(x_1, \ldots, x_n)$, etc.

1. The ring of integers $\mathbb{Z}$:

Gap code
```
gap> LoadPackage( "RingsForHomalg" );
true
gap> ZZ := HomalgRingOfIntegersInSingular( );
Z
gap> a := 4 / ZZ;
4
gap> a^2;
16
```

2. The residue class ring $\mathbb{Z}/\mathbb{Z}4$:

Gap code
```
gap> ZZ_4 := ZZ / 4;
Z/( 4 )
gap> s := 5 / ZZ_4;
|[ 5 ]|
gap> t := 1 / ZZ_4;
|[ 1 ]|
gap> s = t;
true
```

3. The field of rationals $\mathbb{Q}$ and the polynomial ring $\mathbb{Q}[x, y, z]$:

Gap code
```
gap> Q := HomalgFieldOfRationalsInSingular( );
Q
gap> R := Q * "x,y,z";
Q[x,y,z]
```

4. The field of fractions $\mathbb{Q}(x, y, z)$:

Gap code
```
gap> K := AddRationalParameters( Q, "x,y,z" );
Q(x,y,z)
```

6. Now let us define polynomials in $\mathbb{Q}[x, y, z]$:

Gap code
```
gap> f := "x-3*y+2*z^3" / R;
2*z^3+x-3*y
gap> g := "5*y^3+z" / R;
5*y^3+z
gap> h := f * g;
10*y^3*z^3+5*x*y^3-15*y^4+2*z^4+x*z-3*y*z
```

f and g can also be created as follows:

Gap code
```
gap> AssignGeneratorVariables( R );
#I  Assigned the global variables [ x, y, z ]
gap> f := x-3*y+2*z^3;
2*z^3+x-3*y
gap> g := 5*y^3+z;
5*y^3+z
gap> h := f*g;
10*y^3*z^3+5*x*y^3-15*y^4+2*z^4+x*z-3*y*z
```

`homalg` equips all polynomial rings with $>_{\text{drlex}}$ by default:

Gap code
```
gap> LeadingMonomial( h );
y^3*z^3
gap> LeadingCoefficient( h );
10
```

5. The residue class ring $\mathbb{Q}[x, y, z, t]/\langle x - y, z - t\rangle$:

Gap code
```
gap> S := Q * "x,y,z,t";
Q[x,y,z,t]
gap> S := S / [ "x-y", "z-t" ];
Q[x,y,z,t]/( x-y, z-t )
gap> AssignGeneratorVariables( S );
#I  Assigned the global variables [ x, y, z, t ]
gap> x = y;
true
```

Defining matrices in `homalg` can easily be done using the command `HomalgMatrix`. To illustrate this, let us define the matrix

$$M = \begin{bmatrix} x & y & 2 \\ y & 0 & -yz \end{bmatrix}$$

over the ring $R = \mathbb{Q}[x, y, z]$ that we already have defined:

Gap code
```
gap> m := HomalgMatrix( "[ x, y, 2 ,  y, 0, -y*z ]", 2, 3, R );
<A 2 x 3 matrix over an external ring>
gap> Display( m );
x,y,2,
y,0,-y*z
gap> NrColumns( m );
3
gap> NrRows( m );
2
gap> MatElm( m, 2, 3 );
-y*z
gap> HomalgRing( m );
Q[x,y,z]
```

In the following we learn some other basic commands in `homalg`, for example

- `HomalgZeroMatrix(m,n,R)` which returns the zero matrix of dimension $m \times n$ over the given ring R, let us for example construct the zero matrix of dimension 10×2 over the previous ring:

  Gap code
  ```
  gap> z := HomalgZeroMatrix( 10, 2, R );
  <An unevaluated 10 x 2 zero matrix over an external ring>
  gap> Display( z );
  ```

- `HomalgIdentityMatrix(m,R)` which returns the identity matrix over R, let us construct the identity matrix of dimension 2×2 over R.

  Gap code
  ```
  gap> i := HomalgIdentityMatrix( 2, R );
  <An unevaluated 2 x 2 identity matrix over an external ring>
  ```

- `HomalgVoidMatrix([m,][n,]R)` which returns a void matrix, i.e., its content is not yet specified.

  Gap code
  ```
  gap> T1 := HomalgVoidMatrix( R );
  <A void matrix over an internal ring>
  gap> T2 := HomalgVoidMatrix( R, 2, 3 );
  <A void 2 x 3 matrix over an internal ring>
  ```

- `BasisOfRows(A)` which computes the reduced Gröbner basis of $\{f_1, \ldots, f_r\} \subset R^{1\times m}$ (w.r.t. $>_{\text{drlex}}$, term-over-position), where `A` is an $r \times m$ `homalg` matrix whose rows are the f_i's. Let us illustrate this by solving Exercise 3 of Sheet 3 in GAP:

Gap code
```
gap> R := Q * "x,y";
Q[x,y]
gap> m := HomalgMatrix( "[ x^2*y-y^3, x^2, x^3, y ]", 2, 2, R );
<A 2 x 2 matrix over an external ring>
gap> b := BasisOfRows( m );
<A non-zero 5 x 2 matrix over an external ring>
gap> Display( b );
x^2*y-y^3,x^2,
x^3,y,
x*y^3, -x^3+y^2,
y^5,-x^4-x^2*y^2+x*y^2,
0,  x^5-x^2*y^2+y^4
```

- `SyzygiesOfRows(A)` which returns a matrix of row syzygies of A.

Gap code
```
gap> sy_m := SyzygiesOfRows( m );
<An unevaluated 0 x 2 zero matrix over an external ring>
gap> Display( sy_m );
(an empty 0 x 2 matrix)
gap> sy_b := SyzygiesOfRows( b );
<A non-zero 3 x 5 matrix over an external ring>
gap> Display( sy_b );
x,  -y, 1,0,0,
y^2,0,  -x,  1,0,
0,  y^3,-x^2,0,-1
gap> c := sy_b*b;
<An unevaluated 3 x 2 matrix over an external ring>
gap> IsZero( c );
true
```

- `DecideZeroRows(B,A)` (see lecture notes).

- `DecideZeroRowsEffectively(B,A,T)` where `B, A` are `homalg` matrices with the same number of columns and `T` is a void `homalg` matrix. The command returns the matrix $\mathtt{B}' := \mathtt{DecideZeroRows}(\mathtt{B},\mathtt{A})$ and assigns the void matrix `T` such that $\mathtt{B}' = \mathtt{B} + \mathtt{TA}$.

- `RightDivide(B,A)` returns $\mathtt{X}(:= -\mathtt{T})$ such that $\mathtt{XA} = \mathtt{B}$ if the equation $\mathtt{XA} = \mathtt{B}$ is solvable. Otherwise it returns `fail`. Let us examine the solvability of the inhomogeneous linear system of equations $\mathtt{XA} = \mathtt{B}$ in $\mathbb{Q}[x, y, z]$ for

$$\mathtt{A} = \begin{bmatrix} x & y & 2 \\ y & 0 & -yz \end{bmatrix}, \mathtt{B} = \begin{bmatrix} x^2+y^2 & xy & -y^2z+2x \\ xy+yz & y^2 & -yz^2+2y \\ xy+xz & yz & -xyz+2z \\ 0 & y^2 & xyz+2y \end{bmatrix}$$

Gap code

```
gap> R := Q * "x,y,z";
Q[x,y,z]
gap> A:= HomalgMatrix( "[ \
> x,y,2,\
> y,0,-yz ]", 2, 3, R );
<A 2 x 3 matrix over an external ring>
gap> B := HomalgMatrix( "[ \
> x2+y2,xy,-y2z+2x,\
> xy+yz,y2,-yz2+2y,\
> xy+xz,yz,-xyz+2z,\
> 0,y2,xyz+2y ]", 4, 3, R );
<A 4 x 3 matrix over an external ring>
gap> Display( A );
x,y,2,
y,0,-y*z
gap> Display( B );
x^2+y^2,x*y,-y^2*z+2*x,
x*y+y*z,y^2,-y*z^2+2*y,
x*y+x*z,y*z,-x*y*z+2*z,
0,      y^2,x*y*z+2*y
gap> DecideZeroRows( B, A );
<A 4 x 3 zero matrix over an external ring>
gap> T := HomalgVoidMatrix( R );
<A void matrix over an external ring>
gap> DecideZeroRowsEffectively( B, A, T );
<A 4 x 3 zero matrix over an external ring>
gap> Display( T );
-x,-y,
-y,-z,
-z,-x,
-y,x
gap> XX := RightDivide( B, A );
<An unevaluated 4 x 2 matrix over an external ring>
gap> Display( XX );
x,y,
y,z,
z,x,
y,-x
```

```
gap> XX * A = B;
true
```

- `UnionOfRows(A,B)` returns the block matrix $\binom{A}{B}$.
- `UnionOfColumns(A,B)` returns the block matrix $(A\ B)$. Let us construct the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$:

Gap code
```
gap> u1 := UnionOfColumns( A, HomalgZeroMatrix( 2, 3, R ) );
<An unevaluated 2 x 6 matrix over an external ring>
gap> u2 := UnionOfColumns( HomalgZeroMatrix( 4, 3, R ), B );
<An unevaluated 4 x 6 matrix over an external ring>
gap> u := UnionOfRows( u1, u2 );
<An unevaluated 6 x 6 matrix over an external ring>
gap> Display( u );
x,y,2,    0,       0,   0,
y,0,-y*z,0,        0,   0,
0,0,0,    x^2+y^2,x*y,-y^2*z+2*x,
0,0,0,    x*y+y*z,y^2,-y*z^2+2*y,
0,0,0,    x*y+x*z,y*z,-x*y*z+2*z,
0,0,0,    0,       y^2,x*y*z+2*y
gap> v := DiagMat( [ A, B ] );
<An unevaluated 6 x 6 matrix over an external ring>
gap> Display( v );
```

- `CertainRows(A,l)` and `CertainColumns(A,l)` where `A` is `homalg` matrix and `l` is a list, are used to extract specific rows or columns.

Gap code
```
gap> B;
<An unevaluated 4 x 3 matrix over an external ring>
gap> Display( B );
x^2+y^2,x*y,-y^2*z+2*x,
x*y+y*z,y^2,-y*z^2+2*y,
x*y+x*z,y*z,-x*y*z+2*z,
0,      y^2,x*y*z+2*y
gap> r := CertainRows( B, [ 4, 1 ] );
<An unevaluated 2 x 3 matrix over an external ring>
gap> Display( r );
0,      y^2,x*y*z+2*y,
x^2+y^2,x*y,-y^2*z+2*x
gap> c := CertainColumns( B, [ 3 ] );
<An unevaluated 4 x 1 matrix over an external ring>
gap> Display( c );
-y^2*z+2*x,
-y*z^2+2*y,
```

```
-x*y*z+2*z,
x*y*z+2*y
```

- `SyzygiesOfRows(A,N)`, `SyzygiesOfColumns(A,N)`, `DecideZeroColumns(B,A)`, `DecideZeroColumnsEffectively(B,A,T)`, `LeftDivide(B,A)`, `Involution(A)` and a lot more can be found here

  `www.gap-system.org/Manuals/pkg/MatricesForHomalg/doc/chap0.html`

Exercise 2. (Finite left presentations, 16 points)
Let $R = \mathbb{Q}[x, y, z]$ and consider the category R-fpres of finite left presentations over R. Let

$$M = \begin{bmatrix} x & y & 0 \\ z & -x & y \end{bmatrix}, N = \begin{bmatrix} x+y & x-y \\ z & y \\ 0 & -x \\ 0 & z \end{bmatrix}, A = \begin{bmatrix} x+y & 0 \\ x+y & -x \\ -x-y-z & 2x \end{bmatrix}$$

$$B = \begin{bmatrix} -x^2-xy-yz+x+y & -2x^2+xy-y^2-z^2 \\ -xz+x+y & -2xy-xz-x \\ -xz-yz-x-y-z & -2xz+yz+2x \end{bmatrix}$$

$$C = \begin{bmatrix} x^2+xy+yz & -xy^3+x^2-xy+y^2+z^2 \\ xz & 2xy+xz \\ -x^2-xy+xz+yz & -x^2+xy+2xz-yz \end{bmatrix}$$

$$D = \begin{bmatrix} xy+yz & -y & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}, F = \text{The empty matrix of dimensions } 0 \times 1$$

$$G = \begin{bmatrix} x & y \end{bmatrix}, H := \begin{bmatrix} x^2+yz & -xy \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Show that $\varphi := (M, A, N)$ and $\psi := (M, B, N)$ are morphisms in R-fpres and that $\varphi \sim \psi$.
2. Show that (M, C, N) is the zero morphism between M and N.
3. Show that E is a zero object in R-fpres.
4. Compute $K := \mathrm{Ker}(\phi)$ and $\kappa := \mathrm{KernelMono}(\phi)$.
5. Let $\tau := (F, D, M)$. Show that $\tau\phi \sim 0$ and compute the morphism that lifts τ to κ.
6. Compute $\mathrm{Coker}(\phi)$ and $\mathrm{CokernelEpi}(\phi)$.
7. Show that ϕ is not isomorphism.

8. Compute the image mono and co-image epi of ϕ.

9. Compute the isomorphism between image and co-image of ϕ.

10. Give a method to construct equalizers in pre-additive categories with kernels.

11. Let $\alpha := (G, I, M), \beta := (H, J, M)$, compute the categorical intersection of the sub-objects $G \stackrel{\alpha}{\to} M \stackrel{\beta}{\leftarrow} H$.

Hand in until Juli 4th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.