Assignment sheet 7

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Exercise 1. (pre-Abelian but not Abelian, 4 points)
Show that the full subcategory (tfAb) $\subset$ (Ab) of torsion-free Abelian groups is pre-Abelian but not Abelian.

Exercise 2. (Affine rings, 4 points)
Let R be a commutative computable ring and $I = \langle a_1, \ldots, a_n \rangle \trianglelefteq R$ an (explicitly) finitely generated ideal of R. Prove that the residue class ring R/I is again computable.

Exercise 3. (Computable rings, 4 points)
An **involution** on a ring R is an anti-isomorphism $\theta : R \to R$ with $\theta^2 = \mathrm{id}_R$, i.e., θ is an isomorphism of the underlying Abelian group $(R, +)$ and $\theta(1) = 1$, $\theta(\theta(a)) = a$, and $\theta(ab) = \theta(b)\theta(a)$ for all $a, b \in R$. Show that the following are equivalent for any ring R with an involution:

- R is left computable.
- R is right computable.

Exercise 4. (Gaussian normal form algorithm, 4 points)
Given a constructive field k equipped with the Gaussian normal form algorithm, more precisely, an algorithm to compute the row reduced echelon form (RREF). Apply such an algorithm to the block matrix $\left(\begin{smallmatrix} 1 & \mathtt{B} & 0 \\ 0 & \mathtt{A} & 1 \end{smallmatrix}\right)$ and obtain $\left(\begin{smallmatrix} 1 & \mathtt{B}' & -\mathtt{X} \\ 0 & \mathtt{A}' & \mathtt{Y} \\ 0 & 0 & \mathtt{S} \end{smallmatrix}\right)$. Show that setting

- `DecideZeroRows(B, A)` $:= \mathtt{B}'$,
- `DecideZeroRowsEffectively(B, A)` $:= (\mathtt{B}', -\mathtt{X})$,
- `SyzygiesOfRows(A)` $:= \mathtt{S}$,

turn k in to a computable ring. For $k = \mathbb{Q}$ use the previous setting to solve the linear system $XA = B$ where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 28 & 42 \\ 40 & 60 \\ 46 & 69 \end{bmatrix}.$$

Remark. You can use the command `RREF` in Gap, see

`https://www.gap-system.org/Manuals/doc/ref/chap24.html`

Hand in until Juni 27th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.