Assignment sheet 6

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Exercise 1. (Mono, epi, split mono, split epi, 4 points)
Prove that in any category, the following hold:

1. A split epi is an epi.
2. A split mono is a mono.
3. The pre- and post-inverse of an isomorphism coincide, and are hence unique. We call it **the inverse**.
4. The composition of two (split) monos is a (split) mono.
5. The composition of two (split) epis is a (split) epi.
6. The composition of isomorphisms is an isomorphism.
7. If $\phi\psi$ is a mono then ϕ is a mono. The converse is false.
8. If $\phi\psi$ is an epi then ψ is an epi. The converse is false.

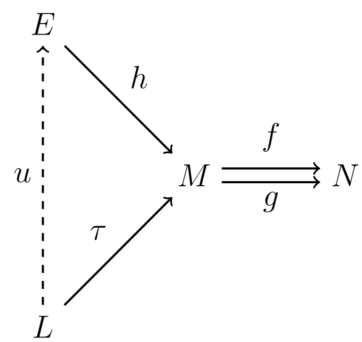
In the category of Abelian groups, show that in the sequence of $\mathbb{Z}$-maps $\mathbb{Z} \overset{2}{\hookrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ the left morphism is mono, the right is epi, and non of them is split (as $\mathbb{Z}$-maps). However, if we view them as maps the category of sets, then both become split.

Exercise 2. (Morphism from coimage to image, 4 points)
Using the notations in Definition 2.22 show that

$$(\epsilon_\kappa \backslash \varphi)/\kappa_\epsilon = \epsilon_\kappa/(\varphi \backslash \kappa_\epsilon).$$

Exercise 3. (Equalizers, 4 points)
Let $\mathcal{A}$ be a category, $M, N \in \mathcal{A}_0$ and $f, g : M \to N \in \mathcal{A}_1$. A pair $E \in \mathcal{A}_0$ and $h : E \to M \in \mathcal{A}_1$ is called **equalizer** of the pair (f, g) if the following two properties hold:

1. $hf = hg$.
2. Given any object L and $\tau : L \to M$ with $\tau f = \tau g$, then there exists a unique morphism $u : L \to E$ such that $uh = \tau$.



The category $\mathcal{A}$ is said to have equalizers if any pair of morphisms $f, g : M \to N \in \mathcal{A}_1$ has an equalizer. Show the following:

1. In any category, if $(E, h : E \to M)$ is an equalizer for a pair $(f, g : M \to N)$, then the morphism h is a mono.
2. If a category has products and equalizers, then it has pullbacks.
3. If a category has a zero object and equalizers, then it has kernels.
4. Define the dual notion of equalizers.

Exercise 4. (Mono, epi in pre-Abelian categories, 4 points)
Let $\mathcal{A}$ be a pre-Abelian category. Prove that for an $\mathcal{A}$-morphism φ:

1. φ is mono iff $\ker \varphi = 0$.
2. φ is epi iff $\operatorname{coker} \varphi = 0$.

Hand in until Juni 20th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.