Assignment sheet 5

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Exercise 1. (Monomial orders, 4 points)
Let $R = k[x_1, \ldots, x_n]$. We found in remark 1.15. in the lecture that

1. the key property of $>_{\text{drlex}}$ is: $>_{\text{drlex}}$ is degree-compatible, and if $f \in R$ is homogeneous, then
$$>_{\text{drlex}} \text{ chooses the leading term of } f \text{ in a subring } k[x_1, \ldots, x_\ell]$$
$$\text{such that } \ell \text{ is as small as possible.}$$

2. the key property of $>_{\text{lex}}$ is that the following holds for all $f \in R$:
$$\mathrm{L}(f) \in k[x_{\ell+1}, \ldots, x_n] \text{ for some } \ell \implies f \in k[x_{\ell+1}, \ldots, x_n].$$

Show that the key properties of $>_{\text{lex}}$ respectively $>_{\text{drlex}}$ characterize these orders among all global monomial orders.
Hint: If $>$ satisfies the key property of $>_{\text{drlex}}$, we have, for instance, $x_2^2 > x_1x_3$. Since $>$ is compatible with multiplication, also $x_2^2x_4 > x_1x_3x_4$.

Exercise 2. (Intersection of ideals, 4 points)
Let $R = k[x_1, \ldots, x_n]$ and I, J be two ideals in R with
$$I = \langle f_1, \ldots, f_s \rangle, J = \langle g_1, \ldots, g_r \rangle.$$

Condider the ideal $L = \langle tf_1, \ldots, tf_s, (1-t)g_1, \ldots, (1-t)g_r \rangle \subseteq R[t]$.

1. For all $r \in R$, show that the *evaluation* map $\phi_r : R[t] \to R; h \mapsto h(r)$ is ring homomorphism.
2. Show that $I \cap J = L \cap R$.
3. How can this result be extended to an algorithm to compute a generating set for $I \cap J$.

Hint: If $h \in R$ then $\phi_0(h) = h = \phi_1(h)$.

Exercise 3. (Subalgebra Membership, 4 points)
With notation as in proposition 1.55, let $\overline{g}, \overline{f}_1, \ldots, \overline{f}_m$ be elements R/I, and let $>$ be a global elimination order on $k[\mathbf{x}, \mathbf{y}]$ with respect to $x_1, \ldots, x_n$. Show:

1. We have $\overline{g} \in k[\overline{f}_1, \ldots, \overline{f}_m]$ iff the normal form $h = \mathrm{NF}(g, J) \in k[\mathbf{x}, \mathbf{y}]$ is contained in $k[\mathbf{y}]$. In this case, $\overline{g} = h(\overline{f}_1, \ldots, \overline{f}_m)$ is a polynomial expression for $\overline{g}$ in terms of the $\overline{f}_k$.

2. The homomorphism $\phi : k[y_1, \dots, y_m] \to R/I$ is surjective iff $\mathrm{NF}(x_i, J) \in k[\mathbf{y}]$ for $i = 1, \dots, n$.

Exercise 4. (Algebra relations, 4 points)

1. Compute the algebra relations on the polynomials

$$f_1 = x^2 + y^2,\ f_2 = x^2y^2,\ f_3 = x^3y - xy^3 \in k[x, y].$$

2. Consider the polynomials

$$g = x^4 + y^4,\ g_1 = x + y,\ g_2 = xy \in k[x, y].$$

   Show that g is contained in the subalgebra $k[g_1, g_2] \subset k[x, y]$, and express g as a polynomial in g_1, g_2.

3. Consider the endomorphism ϕ of $k[x_1, x_2, x_3]$ defined by

$$x_1 \mapsto x_2x_3,\ x_2 \mapsto x_1x_3,\ x_3 \mapsto x_1x_2.$$

   Prove that ϕ induces an automorphism of

$$k[x_1, x_2, x_3]/\langle x_1x_2x_3 - 1\rangle.$$

Remark. You can use Singular to do all computations here. For example, let $R = \mathbb{Q}[x, y]$ and $I = \langle x^2 + y^2x, xy - y^5\rangle$ and let us compute $\mathrm{NF}(y^9 + y^7 + x, I)$ w.r.t. $>_{\mathrm{lex}}$:

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> ring R=0,(x,y),lp;
> poly f1 = x^2+y^2*x;
> poly f2 = x*y-y^5;
> ideal I = f1, f2;
> // Now we make sure that I has a standard basis, i.e., the reduced Gröbner basis
> // otherwise, we may get the rest but not as a k-linear combination in the
> // standard monomials of I
> I = std( I );
> I;
I[1]=y9+y7
I[2]=xy-y5
I[3]=x2+xy2
> reduce( y^9+y^7+x, I );
x
>
```

Hand in until Juni 6th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.