Assignment sheet 4

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Exercise 1. (Syzygy matrices, 4 points)
Let $R = \mathbb{Q}[x,y]$ and $f = x^5 + x^4y + x^3y^2 - x^2y^2 - xy^3 - y^4$, $g = x^5 - 2x^4y + x^3y^2 - x^2y^2 + 2xy^3 - y^4 \in R$.

1. Compute the syzygy matrix of f, g.
2. Is $h = x^6 + 3x^4y^2 - x^3y^3 - x^3y^2 - 3xy^4 + y^5 \in \langle f, g\rangle_R$? If yes then write h as an R-linear combination of f, g.

Exercise 2. (Equality of submodules, 4 points)
Let F be a free R-module, and let $I, J \subset F$ be submodules. Show that if $>$ is any global monomial order on F, then

$$I \subset J \text{ and } \mathrm{L}(I) = \mathrm{L}(J) \implies I = J.$$

Exercise 3. (Greatest common divisor, 4 points)
Let R be a ring. An element c is a greatest common divisor of $a, b \in R$, iff

1. c is a common divisor of a and b, i.e., c divides both a and b,
2. every other common divisor d of a and b divides c.

In all unique factorization domains (e.g. polynomial rings over a field), any two elements have a greatest common divisor. Moreover, in these rings the greatest common divisor of any two element is unique up to a multiplication by a unit. Let $R = k[x_1, \dots, x_n]$ and f, g be two polynomials in R and let $h = \mathrm{GCD}(f, g)$ be the unique monic greatest common divisor of f, g in R.

1. Suppose that $h = \mathrm{GCD}(f, g)$ and $f = f'h, g = g'h$. Show that the syzygy module of f, g is generated by $(-g' \;\; f') \in R^{1\times 2}$.
2. Compute $\mathrm{GCD}(f, g)$ in exercise 1.
3. Give an example of a ring that does not admit greatest common divisors.

Exercise 4. (Hilbert syzygy theorem, 4 points)
Following the matrix version of Hilbert's Syzygy theorem, compute all successive syzygy matrices of the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

Hand in until May 30th 10:30 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.