Assignment sheet 3

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Exercise 1. (Syzygy matrices, 3 points)
Determine a syzygy matrix of $x, y, z \in k[x, y, z]$.

Exercise 2. (Reduced Gröbner basis, 4 points)
Show that if $\langle 0 \rangle \neq I \subset F$ is a submodule, there is a uniquely determined reduced Gröbner basis for I with respect to the given monomial order, namely

$$m_1 - \mathrm{NF}(m_1, I), \ldots, m_r - \mathrm{NF}(m_r, I),$$

where $m_1, \ldots, m_r$ are the minimal generators of $\mathrm{L}(I)$. Explain how to compute the reduced Gröbner basis from any given Gröbner basis.

Exercise 3. (Submodule membership, 4 points)
Let $R = \mathbb{Q}[x, y]$ and $F := R^2$ be the free module over R of rank 2. Let $M := \langle (x^2y - y^3, x^2), (x^3, y) \rangle_R \subset F$.

1. Compute the reduced Gröbner basis G of M w.r.t. an order defined by extending the monomial order $>_{\mathrm{drlex}}$ in R to F.
2. Prove or disprove that $(y^5, x^5 - x^4 - 2x^2y^2 + y^4 + xy^2) \in M$.
3. Compute a syzygy matrix of G.

Exercise 4. (Binomial ideals, 5 points)
A polynomial in R is called a binomial if it has at most two terms. An ideal of R is called a binomial ideal if it is generated by binomials.
Now let $>$ be any global monomial order on R. Show that the following conditions on an ideal $I \trianglelefteq R$ are equivalent:

1. I is a binomial ideal.
2. I has a binomial Gröbner basis, that is, a Gröbner basis consisting of binomials.
3. The normal form mod I of any monomial is a term.

Hand in until May 23th 12:15 in the class or in the Box in ENC, 2nd floor, at the entrance of the building part D.