Assignment sheet 1

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Exercise 1. (Gordan's Lemma, 4 points)
Let k be a field and $R := k[x_1, \dots, x_n]$ be the polynomial ring with indeterminates $x_1, \dots, x_n$ over k. Show that any nonempty set X of monomials in R has only finitely many minimal elements in the partial order given by divisibility. Hence, any monomial ideal in R has finitely many monomial generators.

Exercise 2. (Euclidean Division with Remainder, 5 points)
Let k be a field and $R := k[x]$. Let f be a nonzero polynomial in R. Show that

1. For any polynomial $g \in R$, there are uniquely determined polynomials $g_1, h \in R$ such that $g = g_1 f + h$ and $\deg h < \deg f$. Provide an algorithm to compute such g_1, h.
2. If an expression $g = g_1 f + h$ as in 1. is given, show that $\mathrm{GCD}(f, g) = \mathrm{GCD}(f, h)$ (here, GCD refers to the monic greatest common divisor, i.e., the greatest common divisor where the coefficient of the leading term is 1).
3. Provide an algorithm to compute $\mathrm{GCD}(f, g)$ and $s, t \in R$ with
$$\mathrm{GCD}(f, g) = sf + tg.$$
4. Let $g, f_1, \dots, f_r$ be nonzero polynomials in R and let $I := \langle f_1, \dots, f_r \rangle$. Show that $g \in I$ iff $g = g_1 \, \mathrm{GCD}(f_1, \dots, f_r)$ for some $g_1 \in R$. I.e., I is a principal ideal generated by $\mathrm{GCD}(f_1, \dots, f_r)$.
5. Let $k = \mathbb{Q}$, $f_1 = x^5 - x^3 - 2x$, $f_2 = x^6 + x^4 - 3x^3 - 3x$ and $I := \langle f_1, f_2 \rangle$. Show that $x^4 + x^2 \in I$ and $x^4 + x^2 + 1 \notin I$.

Exercise 3. (Weight order, 3 points)
Let $u = (u_1, \dots, u_n)^t$ be a vector in $\mathbb{R}^{n \times 1}$ such that $u_1, \cdots, u_n$ are positive and linearly independent over $\mathbb{Q}$. Set
$$x^\alpha >_u x^\beta \iff \alpha u > \beta u.$$
Show that $>_u$ is a monomial order.

Exercise 4. (determinate division algorithm, 4 points)
Let $R := \mathbb{Q}[x, y]$ and $f_1 = x^3 - xy + y^2$, $f_2 = x^2 y - y$, $f_3 = xy + y^2$, $g = x^4 - x^2 y \in R$. Using the determinate division algorithm, find $g_1, g_2, g_3, h \in R$ such that
$$g = g_1 f_1 + g_2 f_2 + g_3 f_3 + h,$$
with respect to $>_{\mathrm{lex}}$.

Exercise 5. (Installing Gap/Singular algebra systems)
What is Gap? GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects.
What is Singular? SINGULAR is a computer algebra system for polynomial computations, with special emphasis on commutative and non-commutative algebra, algebraic geometry, and singularity theory.
Installing Gap/Singular. For better and more flexible installation and communication between GAP and SINGULAR, it is highly recommended to install them under Linux or Mac OS X.
GAP installation instructions can be found here

`https://www.gap-system.org/Download/index.html`

SINGULAR installation instructions are presented here

`https://www.singular.uni-kl.de/index.php/singular-download.html`

After installing GAP, make sure that the package IO can be loaded (see below). Installation instructions can be found here

`https://www.gap-system.org/Manuals/pkg/io-4.4.6/doc/chap2.html`

The installation was successful if you run the following commands in GAP and get the following output:

```
gap> 4^2;
16
gap> LoadPackage( "IO" );
true
gap> LoadPackage( "RingsForHomalg" );
true
gap> Q := HomalgFieldOfRationalsInSingular( );
Q
```

Hand in until May 9th 12:00 in the class or in Box in ENC, 2. floor, at the entrance to the building part D.